Part Two

CONTENTS

CHAPTER 1
LIMITS AND CONTINUITY

1.1 Rates of Change and Limits

Average Rates of Change and Secant Lines

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in value of y, $\Delta y = f(x_2) - f(x_1)$, by the length of the interval $\Delta x = x_2 - x_1 = h$ over which the change occurred.

Definition

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

Notice that the average rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$. In geometry, a line joining two points of a curve is called a **secant** to the curve. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ.

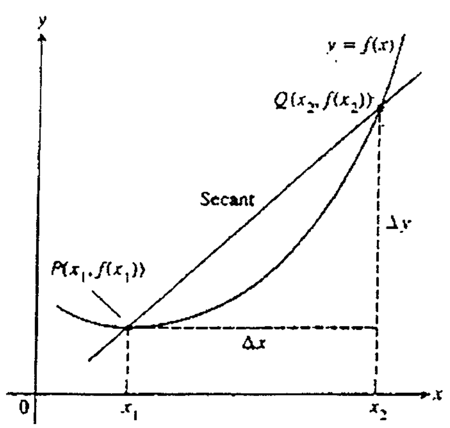


Fig. A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Limits of Function Values

Before we give a definition of limit, let us look at another example.

Example 1 How does the function $f(x)=\dfrac{x^2-1}{x-1}$ behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x=1$ (we cannot divide by zero). For any $x \neq 1$ we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x)=\frac{(x-1)(x+1)}{x-1}=x+1 \text{ for } x \neq 1.$$

The graph of f is thus the line $y = x + 1$ with one point removed, namely the point (1, 2). This removed point is shown as a "hole" in the following figure. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 2 by choosing x *close enough* to 1.

We say that $f(x)$ approaches arbitrarily close to 2 as x approaches 1, or, more simply, $f(x)$ approaches the *limit* 2 as x approaches 1. We write this as

$$\lim_{x\to 1} f(x)=2 \text{ or } \lim_{x\to 1}\frac{x^2-1}{x-1}=2.$$

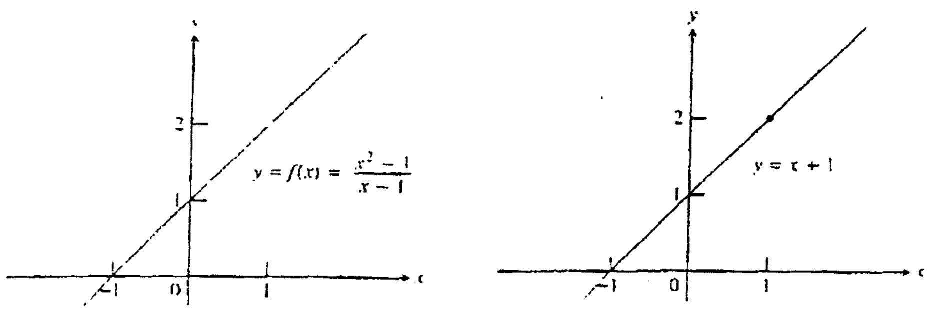


Fig. The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined.

Definition

Informal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x\to x_0} f(x)=L.$$

Example 1

a) $\lim_{x\to 2}(4)=4$

b) $\lim_{x\to -13}(4)=4$

c) $\lim_{x\to 3} x=3$

d) $\lim_{x\to 2}(5x-3)=10-3=7$

e) $\lim_{x \to -2} \frac{3x+4}{x+5} = \frac{-6+4}{-2+5} = -\frac{2}{3}$.

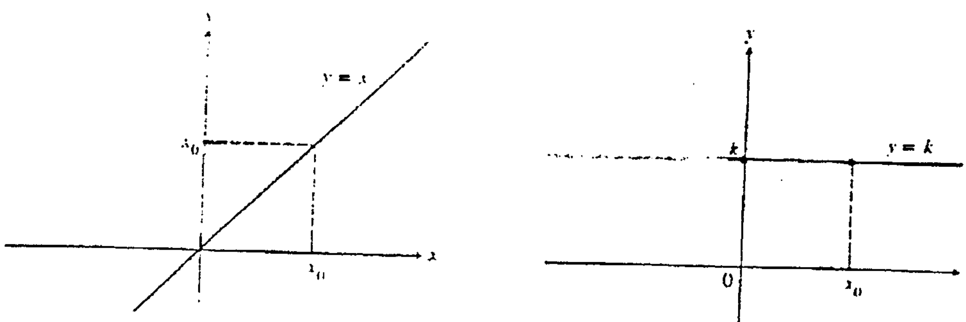
Example 2

a) If f is the **identity function** $f(x) = x$, then for any value of x_0 (Fig a),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0 .$$

b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Fig b),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k .$$



(a) Identity function (b) Constant function

Fig. The functions in Example 2

1.2 Rules for Finding Limits

Limits of Powers and Algebraic Combinations

Theorem 1

Properties of Limits

The following rules hold if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ (L and M real numbers).

1. *Sum Rule*: $\lim_{x \to c} [f(x) + g(x)] = L + M$
2. *Difference Rule*: $\lim_{x \to c} [f(x) - g(x)] = L - M$
3. *Product Rule*: $\lim_{x \to c} f(x) \cdot g(x) = L \cdot M$
4. *Constant Multiple Rule*: $\lim_{x \to c} k\, f(x) = k\,L$ (any number k)
5. *Quotient Rule*: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$
6. *Power Rule*: If m and n are integers, then $\lim_{x \to c} [f(x)]^{m/n} = L^{m/n}$, provided $L^{m/n}$ is a real number.

Example 1 Find $\lim\limits_{x \to c} \dfrac{x^3 + 4x^2 - 3}{x^2 + 5}$.

Solution Starting with the limits $\lim\limits_{x \to c} x = c$ and $\lim\limits_{x \to c} k = k$ from section 1.1, Example 2, and combining them using various parts of Theorem 1, we obtain:

a) $\lim\limits_{x \to c} x^2 = \left(\lim\limits_{x \to c} x\right)\left(\lim\limits_{x \to c} x\right) = c \cdot c = c^2$ Product or Power

b) $\lim\limits_{x \to c}(x^2 + 5) = \lim\limits_{x \to c} x^2 + \lim\limits_{x \to c} 5 = c^2 + 5$ Sum and (a)

c) $\lim\limits_{x \to c} 4x^2 = 4 \lim\limits_{x \to c} x^2 = 4c^2$ Constant Multiple and (a)

d) $\lim\limits_{x \to c}(4x^2 - 3) = \lim\limits_{x \to c} 4x^2 - \lim\limits_{x \to c} 3 = 4c^2 - 3$ Difference and (c)

e) $\lim\limits_{x \to c} x^3 = \left(\lim\limits_{x \to c} x^2\right)\left(\lim\limits_{x \to c} x\right) = c^2 \cdot c = c^3$ Product and (a) or Power

f) $\lim\limits_{x \to c}(x^3 + 4x - 3) = \lim\limits_{x \to c} x^3 + \lim\limits_{x \to c}(4x^2 - 3)$ Sum

$= c^3 + 4c^2 - 3$ (d) and (e)

g) $\lim\limits_{x \to c} \dfrac{x^3 + 4x^2 - 3}{x^2 + 5} = \dfrac{\lim\limits_{x \to c}(x^3 + 4x^2 - 3)}{\lim\limits_{x \to c}(x^2 + 5)}$ Quotient

$= \dfrac{c^3 + 4c^2 - 3}{c^2 + 5}$ (f) and (b)

Example 2 Find $\lim\limits_{x \to -2} \sqrt{4x^2 - 3}$.

$$\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{4(-2)^2 - 3}$$
$$= \sqrt{16 - 3}$$
$$= \sqrt{13}.$$

Two consequences of Theorem 1 further simplify the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial functional as x approaches c, merely substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c at which the *denominator is not zero*, substitute c for x in the formula for the function.

Theorem 2
Limits of Polynomials Can Be Found by Substitution
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \ldots + a_0.$$

Theorem 3
Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 3

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0.$$

Eliminating Zero Denominators Algebraically
Theorem 3 applies only when the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator will sometimes reduce the fraction to one whose denominator is no longer zero at c. When this happens, we can find the limit by substitution in the simplified fraction.

Example 4 *Canceling a common factor*
Evaluate $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$.
Solution We cannot just substitute $x = 1$ because it makes the denominator zero. However, we can factor the numerator and denominator and cancel the common factor to obtain

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}, \quad \text{if } x \neq 1.$$

Thus

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3.$$

Example 5 *Creating and canceling a common factor*
Find $\lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h}$.

Solution We cannot find the limit by substituting $h = 0$, and the numerator and denominator do not have obvious factors. However, we can create a common factor in the numerator by multiplying it (and the denominator) by the so-called *conjugate expression* $\sqrt{2+h}-\sqrt{2}$, obtained by changing the sign between the square roots:

$$\frac{\sqrt{2+h}-\sqrt{2}}{h} = \frac{\sqrt{2+h}-\sqrt{2}}{h}\cdot\frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}$$

$$= \frac{2+h-2}{h(\sqrt{2+h}+\sqrt{2})}$$

$$= \frac{h}{h(\sqrt{2+h}+\sqrt{2})}$$

we have created a common factor of h...

$$= \frac{1}{\sqrt{2+h}+\sqrt{2}}$$

.... which we cancel.

Therefore,

$$\lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h} = \lim_{h\to 0}\frac{1}{\sqrt{2+h}+\sqrt{2}}$$

$$= \frac{1}{\sqrt{2+0}+\sqrt{2}}$$

The denominator is no longer 0 at $h = 0$, so we can substitute.

$$= \frac{1}{2\sqrt{2}}.$$

The Sandwich Theorem

The following theorem will enable us to calculate a variety of limits in subsequent chapters. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L.

Theorem 4

The Sandwich Theorem

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

Example 6 Given that

$$1-\frac{x^2}{4} \le u(x) \le 1+\frac{x^2}{2} \qquad \text{for all } x \ne 0,$$

find $\lim_{x\to 0} u(x)$.

Solution Since

$$\lim_{x\to 0} (1-(x^2/4)) = 1 \text{ and } \lim_{x\to 0} (1+(x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$.

Example 7 Show that if $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$.

Solution Since $-|f(x)| \le f(x) \le |f(x)|$, and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches c, $\lim_{x\to c} f(x) = 0$ by the Sandwich Theorem.

Exercises 1.2

Limit Calculations

Find the limits in Exercises 1-16.

1. $\lim_{x\to -7} (2x+5)$
2. $\lim_{x\to 12} (10-3x)$
3. $\lim_{x\to 2} (-x^2+5x-2)$
4. $\lim_{x\to -2} (x^3-2x^2+4x+8)$
5. $\lim_{t\to 6} 8(t-5)(t-7)$
6. $\lim_{s\to 2/3} 3s(2s-1)$
7. $\lim_{x\to 2} \frac{x+3}{x+6}$
8. $\lim_{x\to 5} \frac{4}{x-7}$
9. $\lim_{y\to -5} \frac{y^2}{5-y}$
10. $\lim_{y\to 2} \frac{y+2}{y^2+5y+6}$
11. $\lim_{x\to -1} 3(2x-1)^2$
12. $\lim_{x\to -4} (x+3)^{1984}$
13. $\lim_{y\to -3} (5-y)^{4/3}$
14. $\lim_{z\to 0} (2z-8)^{1/3}$
15. $\lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1}$
16. $\lim_{h\to 0} \frac{5}{\sqrt{5h+4}+2}$

Find the limits in Exercises 17-30.

17. $\lim_{x\to 5} \frac{x-5}{x^2-25}$
18. $\lim_{x\to -3} \frac{x+3}{x^2+4x+3}$
19. $\lim_{x\to -5} \frac{x^2+3x-10}{x+5}$
20. $\lim_{x\to 2} \frac{x^2-7x+10}{x-2}$

21. $\lim_{t\to 1}\frac{t^2+t-2}{t^2-1}$

22. $\lim_{t\to -1}\frac{t^2+3t+2}{t^2-t-2}$

23. $\lim_{x\to -2}\frac{-2x-4}{x^3+2x^2}$

24. $\lim_{y\to 0}\frac{5y^3+8y^2}{3y^4-16y^2}$

25. $\lim_{u\to 1}\frac{u^4-1}{u^3-1}$

26. $\lim_{u\to 2}\frac{u^3-8}{u^4-16}$

27. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

28. $\lim_{x\to 4}\frac{4x-x^2}{2-\sqrt{x}}$

29. $\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}$

30. $\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}$

1.3 Extensions of the Limit Concept

In this section we extend the concept of limit to

1. *one-sided limits*, which are limits as x approaches a from the left-hand side or the right-hand side only.
2. *infinite limits*, which are not really limits at all, but provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large, positive or negative.

One-Sided Limits

To have a limit L as x approaches a, a function f must be defined on *both sides* of a, and its values $f(x)$ must approach L as x approaches a from either side. Because of this, ordinary limits are sometimes called **two-sided** limits.

It is possible for a function to approach a limiting value as x approaches a from only one side, either from the right or from the left. In this case we say that f has a **one-sided** (either right-hand or left-hand) limit at a.

Definition

Informal Definition of Right-hand and Left-hand Limits

Let $f(x)$ be defined on an interval (a, b) where $a < b$. If $f(x)$ approaches arbitrarily close to L as x approaches a from within that interval, then we say that f has **right-hand limit** L at a, and we write

$$\lim_{x\to a^+} f(x) = L.$$

Let $f(x)$ be defined on an interval (c, a) where $c < a$. If $f(x)$ approaches arbitrarily close to M as x approaches a from within the interval (c, a), then we say that f has **left-hand limit** M at a, and we write

Example 2 All of the following statements about the function graphed in the following Figure are true.

At $x = 0$: $\lim_{x\to 0^+} f(x) = 1$.

$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. (The function is not defined to the left of $x = 0$.)

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x\to 1^+} f(x) = 1$,

$\lim_{x\to 1} f(x)$ does not exist. (The right-and left-hand limits are not equal.)

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$,

$\lim_{x\to 2^+} f(x) = 1$,

$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. (The function is not defined to the right of $x = 4$.)

At every other point a in [0, 4], $f(x)$ has limit $f(a)$.

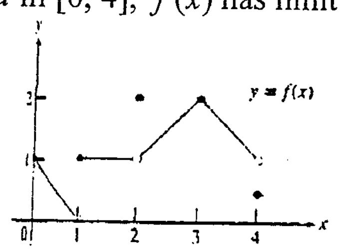


Fig. Graph of the function in Example 2.

Infinite Limits

Let the function $f(x) = 1/x$, $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

In writing this, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim_{x\to 0^+} (1/x)$ *does not exist because* $1/x$ *becomes arbitrarily large and positive as* $x \to 0^+$.

$$\lim_{x \to a^-} f(x) = M.$$

For the function $f(x) = x/|x|$, we have

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1.$$

A function cannot have an ordinary limit at an endpoint of its domain, but it can have a one-sided limit.

Example 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semi-circle in the following Fig. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

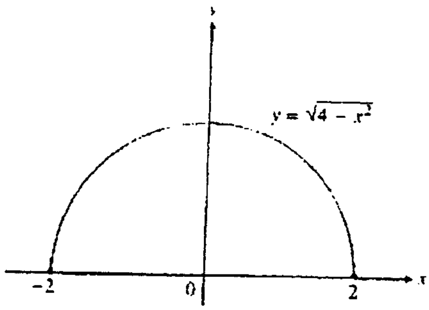


Fig. $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ and $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2.

One-sided limit have all the limit properties listed in Theorem 1. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as does the Sandwich Theorem.

The connection between one-sided and two-sided limits is stated in the following theorem (proved at the end of this section).

Theorem 5

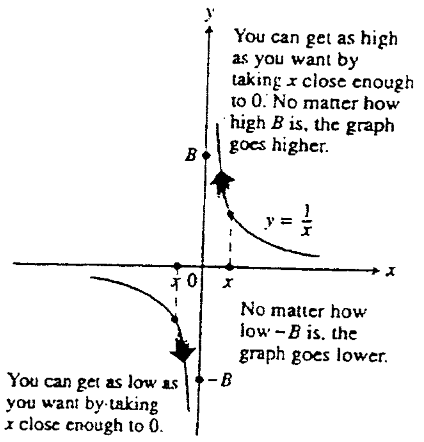
One-sided vs. Two-sided Limits

A function *f(x)* has a limit as x approaches c if and only if it has left-hand and right-hand limits there, and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

As $x \to 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Fig.) We write

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$



1.29 One-sided infinite limits:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There is no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because its values become arbitrarily large and negative.*

Example 3 *One-sided infinite limits*

Find $\lim_{x \to 1^+} \frac{1}{x-1}$ and $\lim_{x \to 1^-} \frac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x - 1)$ is the graph of $y = 1/x$ shifted 1 unit to the right. Therefore, $y = 1/(x - 1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x \to 1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x \to 1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x - 1) \to 0^+$ and $1/(x - 1) \to \infty$. As $x \to 1^-$, we have $(x - 1) \to 0^-$ and $1/(x - 1) \to -\infty$.

Example 4 *Two-sided infinite limits*

Discuss the behavior of

a) $f(x) = \dfrac{1}{x^2}$ near $x = 0$, b) $g(x) = \dfrac{1}{(x+3)^2}$ near $x = -3$.

Solution

a) As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1}{x^2} = \infty .$$

b) The graph of $g(x) = 1/(x+3)^2$ is the graph of $f(x) = 1/x^2$ shifted 3 units to the left. Therefore, g behaves near -3 exactly the way f behaves near 0.

$$\lim_{x\to -3} g(x) = \lim_{x\to -3}\frac{1}{(x+3)^2} = \infty .$$

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to \infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0} (1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0} (1/x^2) = \infty$.

Example 5 *Rational functions can behave in various ways near zeros of their denominators*

a) $\displaystyle\lim_{x\to 2}\frac{(x-2)^2}{x^2-4} = \lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2}\frac{x-2}{x+2} = 0$

b) $\displaystyle\lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2}\frac{1}{x+2} = \frac{1}{4}$

c) $\displaystyle\lim_{x\to 2^+}\frac{x-3}{x^2-4} = \lim_{x\to 2^+}\frac{x-3}{(x-2)(x+2)} = -\infty$

d) $\displaystyle\lim_{x\to 2^-}\frac{x-3}{x^2-4} = \lim_{x\to 2^-}\frac{x-3}{(x-2)(x+2)} = \infty$

e) $\displaystyle\lim_{x\to 2}\frac{x-3}{x^2-4} = \lim_{x\to 2}\frac{x-3}{(x-2)(x+2)}$ does not exist.

f) $\displaystyle\lim_{x\to 2}\frac{2-x}{(x-2)^3} = \lim_{x\to 2}\frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2}\frac{-1}{(x-2)^2} = -\infty .$

Exercises 1.3

Finding Limits Algebraically

Find the limits in Exercises 1-6.

1. $\displaystyle\lim_{x\to -0.5^-}\sqrt{\frac{x+2}{x+1}}$

2. $\displaystyle\lim_{x\to 1^+}\sqrt{\frac{x-1}{x+2}}$

3. $\displaystyle\lim_{x\to -2^+}\left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

4. $\displaystyle\lim_{x\to 1^-}\left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$

5. $\lim_{h \to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$

6. $\lim_{h \to 0^-} \frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}$

1.4 Continuity

Continuity at a Point

In practice, most functions of a real variable have domains that are intervals or unions of separate intervals, and it is natural to restrict our study of continuity to functions with these domains. This leaves us with only three kinds of points to consider: **interior points** (points that lie in an open interval in the domain), **left endpoints**, and **right endpoints.**

Definition

A function f is **continuous at an interior point** $x = c$ of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Definition

A function f is **continuous at a left endpoint** $x = a$ of its domain if

$$\lim_{x \to a^+} f(x) = f(a)$$

and **continuous at a right endpoint** $x = b$ of its domain if

$$\lim_{x \to b^-} f(x) = f(b).$$

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

Example 1 Consider the function $y = f(x)$ in the following Fig., whose domain is the closed interval [0, 4]. Discuss the continuity of f at $x = 0$, 1, 2, 3, and 4.

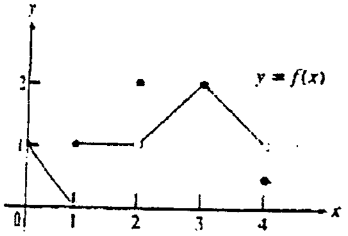


Fig. This function, defined on the closed interval [0,4], is discontinuous at $x = 1$, 2, 3 and 4. It is continuous at all other points of its domain.

Solution The continuity test gives the following results:

a) f is continuous at $x = 0$ because

i) $f(0)$ exists ($f(0) = 1$),

ii) $\lim_{x\to 0+} f(x) = 1$ (the right-hand limit exists at this left endpoint),

iii) $\lim_{x\to 0+} f(x) = f(0)$ (the limit equals the function value).

b) f is discontinuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist. Part 2 of the test fails: f has different right- and left-hand limits at the interior point $x = 1$. However, f is right-continuous at $x = 1$ because

i) $f(1)$ exists ($f(1) = 1$),

ii) $\lim_{x\to 1+} f(x) = 1$ (the right-hand limit exists at $x = 1$),

iii) $\lim_{x\to 1+} f(x) = f(1)$ (the right-hand limit equals the function value).

c) f is discontinuous at $x = 2$ because $\lim_{x\to 2} f(x) \neq f(2)$. Part 3 of the test fails.

d) f is continuous at $x = 3$ because

i) $f(3)$ exists ($f(3) = 2$),

ii) $\lim_{x\to 3} f(x) = 2$ (the limit exists at $x = 2$),

iii) $\lim_{x\to 3} f(x) = f(3)$ (the limit equals the function value).

e) f is discontinuous at the right endpoint $x = 4$ because $\lim_{x\to 4^-} f(x) \neq f(4)$. The right-endpoint version of Part 3 of the test fails.

Theorem 6

Continuity of Algebraic Combinations

If functions f and g are continuous at $x = c$, then the following functions are continuous at $x = c$:

1. $f + g$ and $f - g$
2. fg
3. kf, where k is any number
4. f/g (provided $g(c) \neq 0$)
5. $(f(x))^{m/n}$ (provided $(f(x))^{m/n}$ is defined on an interval containing c, and m and n are integers)

As a consequence, polynomials and rational functions are continuous at every point where they are defined.

Theorem 7

Continuity of Polynomials and Rational Functions

Every polynomial is continuous at every point of the real line. Every rational function is continuous at every point where its denominator is different from zero.

Example 2 The functions $f(x) = x^4 + 20$ and $g(x) = 5x\ (x - 2)$ are continuous at every value of x. The function

$$r(x) = \frac{f(x)}{g(x)} = \frac{x^4 + 20}{5x(x-2)}$$

is continuous at every value of x except $x = 0$ and $x = 2$, where the denominator is 0.

Example 3 *Continuity of* $f(x) = |x|$

The function $f(x) = |x|$ is continuous at every value of x (Fig.). If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \to 0} |x| = 0 = |0|$.

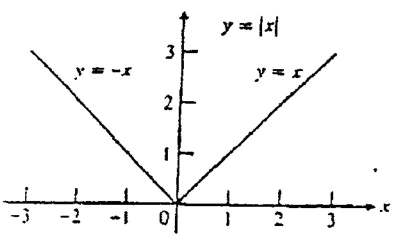


Fig. The sharp corner does not prevent the function from being continuous at the origin (Example 3).

Example 4 *Continuity of trigonometric functions*

The functions sin x and cos x are continuous at every value of x. Accordingly, the quotients

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

are continuous at every point where they are defined.

Exercises 1.4

Continuous Extensions

1. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.
2. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.
3. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.
4. Define $g(4)$ in a way that extends $g(x) = (x^2 - 16)/(x^2 - 3x - 4)$ to be continuous at $x = 4$.

5. For what value of a is

$$f(x)=\begin{cases} x^2-1, & x<3 \\ 2ax, & x\geq 3 \end{cases}$$

continuous at every x?

6. For what value of b is

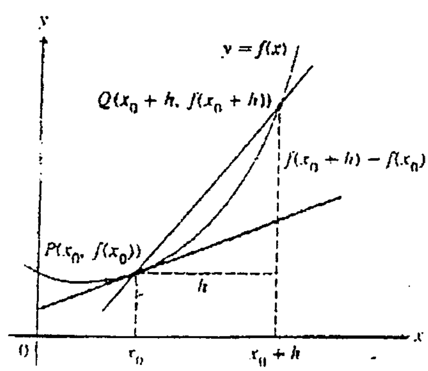
$$g(x)=\begin{cases} x, & x<-2 \\ bx^2, & x\geq -2 \end{cases}$$

continuous at every x?

1.5 Tangent Lines

Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$ we use the same dynamic procedure. We calculate the slope of the secant through P and a point $Q\,(x_0+h, f(x_0+h))$. We then investigate the limit of the slope as $h \to 0$ (Fig.). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.



: 52 The tangent slope is

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

Definitions

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Example 1 *Testing the definition*

Show that the line $y = mx + b$ is its own tangent at any point $(x_0, mx_0 + b)$.

Solution We let $f(x)= mx + b$ and organize the work into three steps.

Step 1: *Find $f(x_0)$ and $f(x_0 + h)$.*

$f(x_0) = mx_0 + b$
$f(x_0 + h) = m(x_0 + h) + b = mx_0 + mh + b$

Step 2: *Find the slope* $\lim_{h\to 0}(f(x_0 + h) - f(x_0))/h$.

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0}\frac{(mx_0+mh+b)-(mx_0+b)}{h}$$

$$= \lim_{h\to 0}\frac{mh}{h} = m$$

Step 3: *Find the tangent line using the point-slope equation.*
The tangent line at the point $(x_0, mx_0 + b)$ is

$y = (mx_0 + b) + m(x - x_0)$
$y = mx_0 + b + mx - mx_0$
$y = mx + b.$

Example 2

a) Find the slope of the curve $y = 1/x$ at $x = a$.
b) Where does the slope equal $-1/4$?
c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0}\frac{-h}{ha(a+h)}$$

$$= \lim_{h\to 0}\frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h\to 0}$" at the beginning of each line until the stage where we could evaluate the limit by substituting $h = 0$.

b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points (2, ½) and (–2, – ½) (Fig. a).

c) Notice that the slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Fig. b).

We see this again as $a \to 0^-$. As a moves away from the origin, the slope approaches 0^- and the tangent levels off.

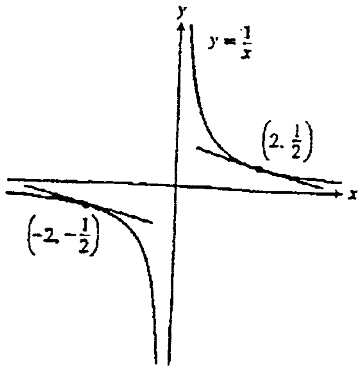


Fig. a The two tangent lines to $y = 1/x$ having slope $-\frac{1}{4}$.

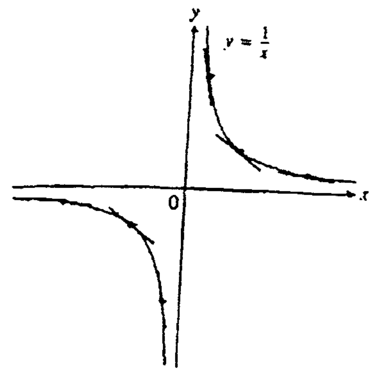


Fig. b The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

Rates of Change

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

is called the **difference quotient of f at x_0**. If the difference quotient has a limit as h approaches zero, that limit is called the **derivative of f at x_0**. If we interpret the difference quotient as a secant slope, the derivative gives the slope of the curve and tangent at the point where $x = x_0$. If we interpret the difference quotient as an average rate of change, the derivative gives the function's rate of change with respect to x at the point $x = x_0$.

All of these refer to the same thing.

1. The slope of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative of f at $x = x_0$
5. $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Example 3 *Instantaneous speed*

We studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t seconds, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. Exactly what was the rock's speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds was

$$\frac{f(1+h)-f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2+2h)}{h} = 16(h+2).$$

The rock's speed at the instant $t = 1$ was

$$\lim_{h\to 0} 16(h+2) = 16(0+2) = 32 \text{ ft/sec.}$$

Our original estimate of 32 ft/sec was right.

Practice Exercises

In Exercises 1-4, find the limit of $g(x)$ as x approaches the indicated values.

1. $\lim_{x\to 0^+} (4g(x))^{1/3} = 2$

2. $\lim_{x\to\sqrt{5}} \frac{1}{x+g(x)} = 2$

3. $\lim_{x\to 1} \frac{3x^2+1}{g(x)} = \infty$

4. $\lim_{x\to -2} \frac{5-x^2}{\sqrt{g(x)}} = 0$

In Exercises 5-12, find the limit or explain why it does not exist.

5. $\lim \frac{x^2-4x+4}{x^3+5x^2-14x}$ (a) as $x \to 0$, (b) as $x \to 2$

6. $\lim \frac{x^2+x}{x^5+2x^4+x^3}$ (a) as $x \to 0$, (b) as $x \to -1$

7. $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

8. $\lim_{x\to a} \frac{x^2-a^2}{x^4-a^4}$

9. $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

10. $\lim_{x\to 0} \frac{(x+h)^2-x^2}{h}$

11. $\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x}$

12. $\lim_{x\to 0} \frac{(2+x)^3-8}{x}$