CHAPTER 4
PERMUTATIONS, COMBINATIONS, AND THE BINOMIAL THEOREM

4.1 Factorial Notation

Definition

(i) $0! = 1.$

(ii) $1! = 1.$

(iii) $n! = 1 \times 2 \times 3 \times \cdots \times (n-1)n$, if n is a positive integer ≥ 2.

Note that $n! = (n-1)!\, n$ for $n \geq 1$.

4.2 Permutations

We consider the number of ways in which we can arrange the letters A, B, and C in a row. There are six different ways:

ABC, ACB, BAC, BCA, CAB, and CBA.

Now consider A, B, C, and D. If we arrange all of them in a row, we have 24 different ways as follows:

$ABCD$, $ABDC$, $ACBD$, $ACDB$, $ADBC$, $ADCB$,
$BACD$, $BADC$, $BCAD$, $BCDA$, $BDAC$, $BDCA$,
$CABD$, $CADB$, $CBAD$, $CBDA$, $CDAB$, $CDBA$,
$DABC$, $DACB$, $DBAC$, $DBCA$, $DCAB$, $DCBA$.

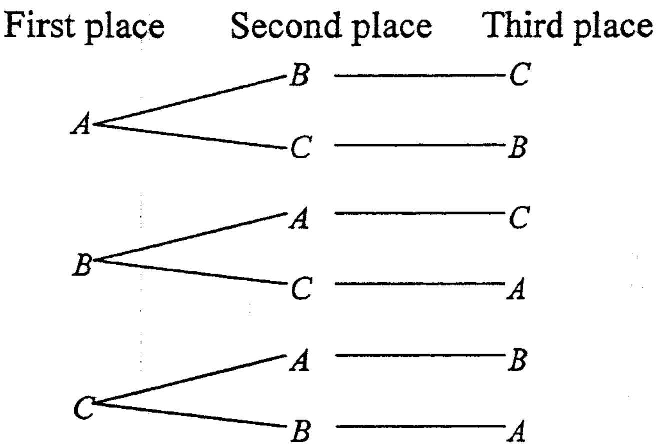
Again we consider the number of arrangements of A, B, C, and D by taking two at time. There are 12 different ways:

$AB, BA, AC, CA, AD, BC, CB, BD, DB, CD, DC.$

Now let us reconsider the first example. In arranging A, B, C, we shall think of as filling three places with three things A, B, C.

The first place can be filled in three ways, with A or B or C. When it is done with either one of these, the second place can be filled in two ways only. When this is done also with one of the two ways, the third place can be filled in only one way. If we multiply these ways 3, 2, 1, together, we get $3 \times 2 \times 1 = 6$.

This gives the same the number of arrangements that was actually seen above. By means of a tree diagram, we have



Similarly, we may explain the other examples.

In the above examples, the number of letters given is small. Hence we can easily write down all possible arrangements. But when we are given a large number of things, it is very time consuming to write down all the possible arrangements. Hence we shall needs methods and theorems for general use.

From the consideration of the above examples we can state the following:

4.3 Fundamental Principle

If one thing can be done independently in n_1 different ways, and if a second thing can be done in n_2 different ways, and a third thing can be done in n_3 different ways, and so on (for any finite number of things), then the total number of ways in which all the things may be done in the stated order is $n_1 n_2 n_3 \dots$.

Definition

Each different arrangement or ordered set of objects is called a *permutation* of those objects.

Notation: The total number of permutations of n objects taken r at a time is denoted by ${}^nP_r$ or $P(n, r)$.

Theorem

$${}^nP_r = n\,(n-1)\,(n-2\,)\,\dots\,(n-r+1).$$

Proof: The possible number of n things taken r at time is equivalent to the number of ways of filling r things in r positions.

There are n ways to fill the first position, $n-1$ ways for the second, $n-2$ ways for the third, etc. The r^{th} position can be filled in $n - (r - 1) = n - r + 1$ ways.

Hence $\quad {}^nP_r = n(n-1)(n-2)\dots(n-r+1).$

Note that

1. ${}^nP_n = n!.$
2. ${}^nP_r = \dfrac{n!}{(n-r)!}.$

Example (4.1)

In how many ways can the letters of the word "Formula" be rearranged?

Solution: "Formula" contains 7 different letters. Total number of arrangements is

$${}^7P_7 = 7! = 5040.$$

The required number = $5040 - 1 = 5039$.

Example (4.2)

How many numbers (each with different digits) between 1000 and 10000 can be formed with 7, 6, 5, 4, 3, 2.

Solution: Each number must contain 4 digits.

The required number = ${}^6P_4 = 6 \times 5 \times 4 \times 3 = 360.$

Example (4.3)

In the arrangements of a, b, c, d, e taken all at a time, how many begin with a?

Solution: Place a in the first position in 1 way. Then we must arrange the remaining 4 letters in ${}^4P_4 = 4! = 24$.

$\therefore$ The required number = $1 \times 24 = 24$.

Example (4.4)

If the letters of the word MEANS are arranged taken all at a time, find how many do not start with AS.

Solution: Total number of arrangements ${}^5P_5 = 5! = 120$.

To find those which begin with AS, place AS in the first two places in 1 way. Then the remaining 3 letters can be arranged in ${}^3P_3 = 3! = 6$ ways.

$\therefore$ The number of arrangements begin with AS $=1\times 6=6$.

$\therefore$ The number of arrangements which do not begin with AS

$$=120-6=114.$$

Example (4.5)

Using the digits of 1, 2, 3, 4, 5, 6, 7, 8, find how many numbers (each of different digits) between 5000 and 6000 can be formed.

Solution: For a number to be between 5000 and 6000, it must begin with 5 and must contain 4 digits.

Place 5 in the first position in 1 way. The remaining 7 digits can be arranged in 3 places in ${}^{7}P_{3}=7\times 6\times 5=210$ ways.

$\therefore$ The required number $=1\times 210=210$.

Example (4.6)

How many arrangements of the letters of the word COINS can be made, without changing the place of the vowels in the word?

Solution: O and I must be fixed. The consonants can be arranged in ${}^{3}P_{3}=3!=6$ ways.

This is the required number of arrangements.

Example (4.7)

Find the number of arrangements of the letters of the word STRANGE (taken all at a time) can be made so that the vowels may be only in the odd places.

Solution: We consider A, E out of 7 places, the odd places are first, third, fifth and seventh places. A, E can be placed ${}^{4}P_{2}=4\times 3=12$ ways.

The remaining five letters can be arranged in ${}^{5}P_{5}=5!=120$ ways.

$\therefore$ The required number $=120\times 12=1440$.

Example (4.8)

If all the letters A, B, C, D, E, F, G are arranged, find the number of arrangements in which

(i) A, B, C, are together,

(ii) A, B, C are not all together.

Solution: (i) Suppose (ABC) is one. We must arrange (ABC), D, E, F, G in ${}^{5}P_{5}=5!=120$ ways.

In each of these ways A, B, C can be arranged in ${}^3P_3 = 3! = 6$ ways among themselves.

$\therefore$ The required number $= 120 \times 6 = 720$.

(ii) The total number of ways of arranging the letters is

$${}^7P_7 = 7! = 5040.$$

$\therefore$ The required number $= 5040 - 720 = 4320$.

Example (4.9)

Find the number of numbers containing 4 digits.

Solution: The first digit must not be 0, but could be any one of 1, 2, 3, 4, 5, 6, 7, 8, 9.

$\therefore$ The first place can be filled in 9 ways. The remaining 3 places can be filled in 10 ways each.

$\therefore$ The required number of numbers $= 9 \times 10 \times 10 \times 10 = 9000$.

4.4 Permutations of *n* Things not All Different

The number of permutations of *n* things taken all at a time, when *r* are of one kind, *s* are of another kind and the rest are different would be

$$\frac{n!}{r!\ s!}.$$

Such formula can be extended as required.

Example (4.10)

Find the number of ways in which the letters of the word INFINITESIMAL can be arranged.

Solution: There are 13 letters, in which we have 4I and 2N. The rest are different. Therefore the required number of permutations

$$= \frac{13!}{4!2!} = 129729600.$$

Example (4.11)

If the letters of the word CONSONANT are arranged taken all at a time, find the number of arrangements in which

(i) the two O's are together,

(ii) the arrangements begin with the three N's.

Solution: (i) If the two O's are regarded as one thing, there are 8 letters.

$$\text{Required number } \frac{8!}{3!} = 6720.$$

(ii) Place three N's at the beginning.

$$\text{Required number } \frac{6!}{2!} = 360.$$

Example (4.12)

By arranging all the digits of the number 242302, find the number of numbers greater than 200000.

Solution: Total number of possible arrangements $= \frac{6!}{3!} = 120$.

From this we must disregard those which begin with 0 and this number $= \frac{5!}{3!} = 20$.

The required number = 120 – 20 = 100.

4.5 Combinations

Definition

A *combination* is a set of objects in no particular order.

Note: The number of combinations of n thing taken r at a time is denoted by ${}^nC_r$ or $C(n, r)$. Here $0 \le r \le n$.

Theorem

$$\begin{aligned} {}^nC_r &= \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} \\ &= \frac{n!}{(n-r)!r!}. \end{aligned}$$

Proof: The objects in each combination can be arranged to give $r!$ different permutations.

$$\therefore {}^nC_r \cdot r! = {}^nP_r.$$

$$\begin{aligned} {}^nC_r &= \frac{{}^nP_r}{r!} \\ &= \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} \\ &= \frac{n!}{(n-r)!r!}. \end{aligned}$$

Note: $^nC_0 = 1, \quad ^nC_n = 1, \quad ^nC_1 = n$.

Corollary: $^nC_r = {}^nC_{n-r}$.

Example (4.13)

If $^nC_{n-4} = 15$, find n.

Solution:

$$^nC_{n-4} = 15$$
$$\therefore {}^nC_4 = 15$$
$$\frac{n(n-1)(n-2)(n-3)}{4!} = 15$$
$$\therefore n(n-1)(n-2)(n-3) = 15 \times 4!$$
$$= 6 \times 5 \times 4 \times 3$$
$$\therefore n = 6.$$

Example (4.14)

If $^{20}C_{r+11} = {}^{20}C_{3r+1}$, find r.

Solution: (i)

$$r + 11 = 3r + 1$$
$$2r = 10$$
$$r = 5.$$

(or) (ii)

$$^{20}C_{r+11} = {}^{20}C_{20-(r+11)}$$
$$= {}^{20}C_{9-r}$$
$$9 - r = 3r + 1$$
$$r = 2.$$

Example (4.15)

If $^nP_r = 720, \quad ^nC_r = 120$, find n and r.

Solution:

$$\frac{^nP_r}{r!} = {}^nC_r$$
$$\therefore \frac{720}{r!} = 120$$
$$r! = 6 = 3!.$$
$$\therefore r = 3.$$
$$\therefore {}^nP_3 = 720$$
$$n(n-1)((n-2) = 10 \times 9 \times 8$$
$$\therefore n = 10.$$

Example (4.16)

How many diagonals has a decagon?

Solution: A decagon has 10 sides. The number of lines joining any two vertices is

$$^{10}C_2 = \frac{10\times 9}{2!} = 45.$$

This includes 10 sides.

$$\therefore \text{ The number of diagonals} = 45 - 10 = 35.$$

Example (4.17)

There are 5 teachers and 10 students. In how many ways can a committee of 3 teachers and 2 students be formed?

Solution: The required number of ways

$$^{5}C_3 \times {}^{10}C_2 = \frac{5\times 4\times 3}{3!}\times\frac{10\times 9}{2!} = 450.$$

Example (4.18)

In how many ways can 4 fruits be selected out of 10 fruits so as to include the largest fruit?

Solution: Select the largest fruit. Then 9 fruits are left from which we must select 3.

$$\text{The number of ways} = {}^{9}C_3 = \frac{9\times 8\times 7}{3!} = 84.$$

Example (4.19)

In how many ways can 4 fruits be selected out of 10 fruits so as to exclude the smallest fruit?

Solution: We must choose from 9 fruits.

$$\text{The number of ways} = {}^{9}C_4 = \frac{9\times 8\times 7\times 6}{4!} = 126.$$

Example (4.20)

A boat's crew consists of 8 men, of whom 2 can only row on one side and 2 only on the other. In how many ways can a selection be made so that 4 men may row on each side?

Solution: The required number $= {}^{4}C_2 = \frac{4\times 3}{2!} = 6.$

Example (4.21)

A given group of 50 people contains 5 teachers. In how many ways can a selection of 5 people be made so as to include at least one teacher?

Solution: Total number of ways of choosing 5 people $={}^{50}C_5$ and

total number of ways in which no teacher is include $={}^{45}C_5$.

$$\therefore \text{ The required number} = {}^{50}C_5 - {}^{45}C_5$$

$$= 2118760 - 1221759 = 897001.$$

Example (4.22)

If an alphabet consists of 5 vowels and 15 consonants, how many words of 3 different vowels and 2 different consonants can be made?

Solution: The vowels and consonants can be selected in ${}^{5}C_3 \times {}^{15}C_2$

$$= \frac{5\times4\times3}{3!} \times \frac{15\times14}{2!} = 1050 \text{ ways.}$$

For selection, the letters can be arranged in ${}^{5}P_5 = 5! = 120$ ways.

Hence the number of words $= 1050 \times 120 = 126000$.

4.6 Binomial Theorem (for a Positive Integral Index)

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_rx^{n-r}y^r + \ldots + {}^nC_ny^n.$$

Here, the $(r+1)^{th}$ term is $U_{r+1} = {}^nC_rx^{n-r}y^r$ and it is called the *general term.*

The right hand side of the above equation is called the *binomial expansion* of $(x+y)^n$.

Note that there are $n + 1$ terms in the expansion.

If we put $x = 1$ and $y = x$ in the above equation, we get

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n.$$

Example (4.23)

Expand $\left(\frac{x}{2}+2y\right)^6$.

Solution:

$$\left(\frac{x}{2}+2y\right)^6 = \left(\frac{x}{2}\right)^6 + {}^6C_1\left(\frac{x}{2}\right)^5(2y) + {}^6C_2\left(\frac{x}{2}\right)^4(2y)^2 + {}^6C_3\left(\frac{x}{2}\right)^3(2y)^3$$

$$+ {}^6C_4\left(\frac{x}{2}\right)^2(2y)^4 + {}^6C_5\left(\frac{x}{2}\right)(2y)^5 + {}^6C_6(2y)^6$$

$$\left(\frac{x}{2}+2y\right)^6=\frac{x^6}{64}+\frac{6x^5}{32}(2y)+\frac{6\cdot5}{1\cdot2}\cdot\frac{x^4}{16}(4y^2)+\frac{6\cdot5\cdot4}{1\cdot2\cdot3}\cdot\frac{x^3}{8}\cdot(8y^3)$$

$$+\frac{6\cdot5}{1\cdot2}\cdot\frac{x^2}{4}\cdot(16y^4)+6\cdot\frac{x}{2}\cdot(32y^5)+64y^6$$

$$=\frac{x^6}{64}+\frac{3}{8}x^5y+\frac{15}{4}x^4y^2+20x^3y^3+60x^2y^4+96xy^5+64y^6.$$

Example (4.24)

Find the tenth term in the expansion of

$$\left(\frac{a}{b}-\frac{2b}{a^2}\right)^{13}.$$

Solution: $U_{r+1}={}^nC_r x^{n-r}y^r$.

Here $x=\frac{a}{b}$, $y=-\frac{2b}{a^2}$,and $n=13$.

$$r+1=10\Rightarrow r=9$$

$$U_{10}={}^{13}C_9\left(\frac{a}{b}\right)^{13-9}\left(\frac{-2b}{a^2}\right)^9=-{}^{13}C_4\,\frac{a^4}{b^4}\cdot\frac{2^9.b^9}{a^{18}}$$

$$=-366080\,\frac{b^5}{a^{14}}.$$

Example (4.25)

Find the middle term in the expansion of

$$\left[\frac{y\sqrt{x}}{3}-\frac{3}{x\sqrt{y}}\right]^{16}.$$

Solution: The number of terms = 16 + 1 = 17.

∴ The middle term is the 9^{th} term.

Hence $$U_9={}^{16}C_8\left(\frac{y\sqrt{x}}{3}\right)^{16-8}\left(\frac{-3}{x\sqrt{y}}\right)^8$$

$$={}^{16}C_8\,\frac{y^8x^4}{3^8}\cdot\frac{3^8}{x^8y^4}$$

$$={}^{16}C_8\,\frac{y^4}{x^4}=12870\,\frac{y^4}{x^4}.$$

Example (4.26)

Find the coefficient of x^{13} in the expansion of

$$(ax - x^2)^{10}.$$

Solution: $U_{r+1} = {}^{10}C_r (ax)^{10-r} (-x^2)^r$

$$= (-1)^r \, {}^{10}C_r \, a^{10-r} \, x^{10-r} \, x^{2r}$$

$$= (-1)^r \, {}^{10}C_r \, a^{10-r} \, x^{10+r}.$$

To get x^{13}, put $10 + r = 13$.

Therefore $r = 3$.

The term containing x^{13} is $U_4 = (-1)^3 \, {}^{10}C_3 \, a^7 \, x^{13}$

$$= -120 \, a^7 \, x^{13}.$$

Therefore the coefficient of x^{13} is $-120 \, a^7$.

Example (4.27)

Find the term independent of x in the expansion of

$$\left(3x - \frac{2}{x^2}\right)^{15}.$$

Solution: $U_{r+1} = {}^{15}C_r (3x)^{15-r} \left(\frac{-2}{x^2}\right)^r$

$$= (-2)^r (3)^{15-r} \, {}^{15}C_r \, \frac{x^{15-r}}{x^{2r}}$$

$$= (-2)^r (3)^{15-r} \, {}^{15}C_r \, .x^{15-3r}.$$

Put $15 - 3r = 0$ (or) $r = 5$.

Then the required term is $U_6 = (-2)^5 (3)^{10} \, {}^{15}C_5$

$$= -\frac{15.14.13.12.11}{5.4.3.2.1} \, 3^{10} \cdot 2^5.$$

4.7 Binomial Theorem (for a Rational Index)

Suppose x is a real number and n is a rational number. Then, for $-1 < x < 1$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots,$$

the RHS being an infinite series.

Note: We can deduce the following useful identities.

1. $\dfrac{1}{1+x}=1-x+x^2-x^3+\dots$
2. $\dfrac{1}{1-x}=1+x+x^2+x^3+\dots$
3. $\dfrac{1}{(1+x)^2}=1-2x+3x^2-4x^3+\dots$
4. $\dfrac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\dots$

Note that these are valid for $-1<x<1$.

Example (4.28)

Find A, B, C such that

$$\frac{2x}{(1-x)(1+x^2)}=\frac{A}{1-x}+\frac{B+Cx}{1+x^2}.$$

Hence obtain the expansion of the fraction in ascending power of x as far as the term in x^5. Between what values must x lie so that the expansion may be valid?

Solution: $$\frac{2x}{(1-x)(1+x^2)}=\frac{A}{1-x}+\frac{B+Cx}{1+x^2}.$$

$$\therefore\ 2x=A(1+x^2)+(B+Cx)(1-x)$$

$$=(A-C)x^2+(C-B)x+(A+B)$$

$$\therefore\ A-C=0, \qquad C-B=2, \text{ and } \qquad A+B=0.$$

$$\therefore\ A=1, \qquad B=-1, \qquad C=1.$$

Therefore

$$\frac{2x}{(1-x)(1+x^2)}=\frac{1}{1-x}+\frac{x-1}{1+x^2}$$

$$=(1+x+x^2+x^3+x^4+x^5+\dots)+(x-1)(1-x^2+x^4-\dots)$$

$$=2x+2x^2+2x^5+\dots$$

For the expansion to be valid, we must have $-1<x<1$ and $-1<x^2<1$.

$\therefore\ -1<x<1$ is the required interval.

Example (4.29)

Find an approximate value of $\sqrt{3.98}$, correct to four decimal places.

Solution: $\sqrt{3.98} = (4-0.02)^{\frac{1}{2}} = 2(1-0.005)^{\frac{1}{2}}$

$$= 2\left[1 + \frac{1}{2}(-0.005) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-0.005)^2}{2!} + \ldots\right]$$

= 1.9950, to 4 decimal places.

Example (4.30)

Find the value of $(1.03)^{\frac{3}{2}} - (0.97)^{\frac{3}{2}}$, correct to five places of decimals.

Solution: $(1.03)^{\frac{3}{2}} = (1+0.03)^{\frac{3}{2}}$

$$= 1 + \frac{3}{2}(0.03) + \frac{\frac{3}{2}.\frac{1}{2}}{2!}(0.03)^2 + \frac{\frac{3}{2}.\frac{1}{2}.\left(-\frac{1}{2}\right)}{3!}(0.03)^3 + \ldots$$

and

$$(0.97)^{\frac{3}{2}} = (1+(-0.03))^{\frac{3}{2}}$$

$$= 1 + \frac{3}{2}(-0.03) + \frac{\frac{3}{2}.\frac{1}{2}}{2!}(-0.03)^2 + \frac{\frac{3}{2}.\frac{1}{2}.\left(-\frac{1}{2}\right)}{3!}(-0.03)^3 + \ldots$$

$$\therefore\ (1.03)^{\frac{3}{2}} - (0.97)^{\frac{3}{2}} = 3(0.03) - \frac{1}{16}(0.03)^3 + \ldots$$

= 0.09000, correct to five decimal places.

Problem Set (4.1)

4.1.1 Show that $\frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} = \frac{50}{7!}$.

4.1.2 How many arrangements of the letters of the word COINS can be made, if every arrangement is to begin and end with a vowel?

4.1.3 In how many ways can the letters of the word EQUATION be arranged?

4.1.4 If ${}^{n}P_5 = {}^{n+1}P_4$, find n.

4.1.5 If the number of permutations of n things taken all at a time equals 5040, find n.

4.1.6 If ${}^{6}P_r = 360$, find r.

4.1.7 If ${}^{m+n}P_2 = 56$, ${}^{m-n}P_2 = 12$, find m and n.

4.1.8 Show that ${}^{n}P_r = {}^{(m-1)}P_r + r.{}^{(m-1)}P_r + r.{}^{n-1}P_{r-1}$.

4.1.9 How may numbers not more than 20 digits are there which can be formed using 0, 1, 2, 3, 4 any number of times? (Ans: 5^{20})

4.1.10 How many words can be formed of the five letters c, r, e, s, t so that the vowel may be the central letter?

4.1.11 How many numbers are there which consist of seven figures (digits)? (Ans: 6×10^6)

4.1.12 How many numbers not more than four digits can be formed with 7, 8, 9? (Ans: 120)

Problem Set (4.2)

4.2.1 How many permutations are there of the letters of the word IRRAWADY?

4.2.2 How many arrangements can be made with the letters of the words
(i) PROPORTION, (ii) DIVISTIONS.

4.2.3 How many permutations can be made of all the letters of the word ENGINEERING? In how many of these will the three E's stand together?

Problem Set (4.3)

4.3.1 A father with x children takes them three at a time to the Zoological Garden as the same three children together not more than once. How often can he go?

4.3.2 How many triangles can be formed with 10 straight lines in a plane, of which no two are parallel, and no three meet in common point?

4.3.3 From a company of 20 soldiers, four are placed on guard every two hours. For what length of time can different sets be selected?

4.3.4 If ${}^{n}C_{20} = {}^{n}C_{35}$, find n.

4.3.5 If ${}^{20}C_{r+4} = {}^{20}C_{r-4}$, find r.

4.3.6 If ${}^{n}C_3 = 5\ {}^{n}C_5$, find n.

4.3.7 If ${}^{n}P_3 = 6 \times {}^{n}C_4$, find n.

Problem Set (4.4)

4.4.1 Find the expansion of $(2x+3y)^5$, $(5+2x^2)^4$, $(3xy-\frac{2x}{y})^5$, $(1-\frac{x}{2})^3$.

4.4.2 Find and simplify the fifth term in the expansion of $(\frac{1}{2}x-2y)^9$.

4.4.3 Find and simplify the two middle terms in the expansion of $(\frac{1}{2}x-2y)^7$.

4.4.4 Find the coefficient of $\frac{1}{x}$ in the expansion of $(\frac{x}{3}-\frac{1}{2x})^{15}$.

4.4.5 Find the term independent of x in the expansion of $(\frac{1}{2}x^2-\frac{1}{x})^6$.

4.4.6 Show that there is no term containing a^{12} in the expansion of $(a^2-\frac{x}{a^3})^{10}$.

4.4.7 Show that ${}^nC_0+{}^nC_1+\ldots+{}^nC_n=2^n$.

4.4.8 Show that ${}^nC_0+{}^nC_2+{}^nC_4+\ldots={}^nC_1+{}^nC_3+\ldots=2^{n-1}$.

Problem Set (4.5)

4.5.1 Find the values of $\sqrt{4.02}, \sqrt{8.96}, \sqrt[5]{31.95}, \frac{1}{\sqrt{9.02}}$ correct to 4 decimal places.

4.5.2 Find the first 4 nonzero terms in the expansion of $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$ in ascending powers of x.

4.5.3 Find A, B, C such that

$$F=\frac{x^2+2x}{(x^2+2)(x-1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2}.$$

Use it to expand F in ascending powers of x as far as the term in x^4. State the necessary restriction on the values of x.

4.5.4 Prove that the first term in the expansion of $\frac{1}{1-x}-\frac{2}{(1-2x)^{1/2}}+\frac{1}{(1-3x)^{1/3}}$ in ascending powers of x is $\frac{2}{3}x^3$. For what values of x in this expansion valid?

4.5.5 Resolve $\dfrac{7-8x}{(1-x)(2-x)}$ into partial fractions. Hence find the expansion of the function in ascending powers of x as far as three nonzero terms. State for which values of x this expansion is valid.

4.5.6 If x is so small that its cube and higher powers can be neglected, prove that $\sqrt{\dfrac{1+x}{1-x}} \simeq 1+x+\dfrac{1}{2}x^2$. By taking $x=\dfrac{1}{9}$, prove that $\sqrt{5}$ is approximately equal to $\dfrac{181}{81}$.

CHAPTER 5
MATRICES AND DETERMINANTS

5.1 Matrices

Definition

A set of *mn* numbers arranged in a rectangular array of *m* rows and *n* columns

$$A=\begin{bmatrix} a_{11} & a_{22} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called a *matrix* of numbers, and a matrix is often more compactly written $A=\left[a_{ij}\right]_{m,n}$.

The numbers $a_{ij}(i=1,2,\cdots,m\ ;\ j=1,2,\cdots,n)$ that comprise the given matrix are called the *entries* (or *elements*) of the matrix.

We say that the matrix A has *dimensions* $m\times n$, or A is an *m*-by-*n* matrix. If $m = n$, the matrix is called a *square matrix* of order *n*. If $m\neq n$, then it is a *rectangular matrix.* A $1\times n$ matrix is called a *row vector* and an $m\times 1$ matrix, a *column vector.* An ordinary number may be regarded as a 1×1 matrix.

A square matrix of the form

$$A=\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

is termed diagonal and briefly denoted as $\left[a_{11},\quad a_{22},\quad \cdots,\quad a_{nn}\right]$.

If $a_{ii}=1$ $(i = 1, 2, \ldots, n)$ a diagonal matrix is called a *unit matrix* and is denoted by *I*; thus

$$I=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

A matrix with all elements zero is called a *zero matrix* and is denoted by $\mathbf{0}$.

5.2 Operations Involving Matrices

(1) Equality of Matrices

The matrices A and B are equal $A = B$, if they have the same dimensions, and the corresponding entries are equal.

(2) The Sum and Difference of Matrices

The sum of two matrices $A= [a_{ij}]$ and $B = [b_{ij}]$ of the same dimensions is a matrix $C = [c_{ij}]$ of the same dimensions with

$$c_{ij} = a_{ij} + b_{ij} .$$

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$,

then

$$C = A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}.$$

The following properties are derived directly from the definition of a matrix sum:

(i) $A + (B + C) = (A + B) + C$,

(ii) $A + B = B + A$,

(iii) $A + \mathbf{0} = A$.

The difference of two matrices is defined analogously:

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{bmatrix}.$$

(3) Multiplication of a Matrix by a Scalar

The product of a matrix $A = [a_{ij}]$ by a scalar α is a matrix whose entries are obtained by multiplying all entries of A by the scalar α; that is,

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}.$$

The following properties are derived directly from the definition of the product of a matrix by a scalar:

(i) $1A = A$,

(ii) $0A = \mathbf{0}$,

(iii) $\alpha(\beta A) = (\alpha\beta)A$,

(iv) $(\alpha + \beta)A = \alpha A + \beta A$,

(v) $\alpha(A + B) = \alpha A + \alpha B$.

Here A, B are matrices, and α, β are scalars.

(4) Multiplication of Matrices

Suppose A and B are matrices of dimensions $m \times p$ and $p \times n$ respectively. They their product AB is a matrix of dimensions $m \times n$.

Example (5.1)

$$\text{Let } A = \begin{bmatrix} 3 & 2 & 8 & 1 \\ 1 & 0 & -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 3(1) + 2(3) + 8(5) + 1(7) & 3(2) + 2(4) + 8(6) + 1(8) \\ 1(1) + 0(3) + (-3)(5) + 4(7) & 1(2) + 0(4) + (-3)(6) + 4(8) \end{bmatrix}.$$

$$= \begin{bmatrix} 56 & 70 \\ 14 & 16 \end{bmatrix}.$$

Example (5.2)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1(1) + 2(2) + 3(3) \\ 4(1) + 5(2) + 6(3) \\ 7(1) + 8(2) + 9(3) \end{bmatrix} = \begin{bmatrix} 14 \\ 32 \\ 50 \end{bmatrix}.$$

Note: The product of two matrices is generally non commutative, $AB \neq BA$.

Example (5.3)

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

Then $AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$, $BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$

and we see that $AB \neq BA$..

Example (5.4)

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 4 & 3 & 0 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 19 & 13 & 7 \\ 46 & 31 & 19 \end{bmatrix}$ but BA does not exist.

5.3 The Transpose of a Matrix

If in an m x n matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

we replace the rows by the columns, we get the transpose of A:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}$$

of dimensions $n \times m$.

The matrix A is called a *symmetric matrix* if $A^T = A$.

The transpose of a matrix has the following properties:

(i) $(A^T)^T = A.$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(AB)^T = B^T A^T$.

Example (5.5)

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

Then $$A^T=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix},\quad B^T=\begin{bmatrix}5 & 7\\ 6 & 8\end{bmatrix},$$

$$A+B=\begin{bmatrix}6 & 8\\ 10 & 12\end{bmatrix},\text{ and } AB=\begin{bmatrix}19 & 22\\ 43 & 50\end{bmatrix}.$$

Therefore $$(A^T)^T=\left(\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}\right)^T=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=A,$$

$$(A+B)^T=\begin{bmatrix}6 & 10\\ 8 & 12\end{bmatrix}=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}+\begin{bmatrix}5 & 7\\ 6 & 8\end{bmatrix}$$

$$=A^T+B^T,$$

and $$(AB)^T=\begin{bmatrix}19 & 43\\ 22 & 50\end{bmatrix}=\begin{bmatrix}5 & 7\\ 6 & 8\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$$

$$=B^TA^T.$$

5.4 Determinants of Order Two and Three

(1) Determinants of Order Two

The symbol $\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}$, consisting of 4 numbers called elements arranged in two rows and two columns, is called a *determinant* of order two. The elements a_1 and b_2 are said to lie along the *principal diagonal*; the elements a_2 and b_1 are said to lie along the *secondary diagonal.*

The value of the determinants is obtained by forming the product of the elements along the principal diagonal and subtracting from it the product of the elements along the secondary diagonal; thus,

$$\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}=a_1b_2-a_2b_1.$$

The solution of the consistent and independent equations

$$a_1x+b_1y=c_1$$
$$a_2x+b_2y=c_2$$

may be expressed as

$$x=\frac{N_x}{D}=\frac{\begin{vmatrix}c_1 & b_1\\ c_2 & b_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}\quad;\quad y=\frac{N_y}{D}=\frac{\begin{vmatrix}a_1 & c_1\\ a_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}.$$

Example (5.6)

Solve $\begin{cases} y = 3x + 1 \\ 4x + 2y - 7 = 0 \end{cases}$

using determinants.

Solution: Arrange the equations in the form

$$3x - y = -1$$

$$4x + 2y = 7.$$

Therefore

$$D = \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} = (3)(2) - (-1)(4) = 10,$$

$$N_x = \begin{vmatrix} -1 & -1 \\ 7 & 2 \end{vmatrix} = (-1)(2) - (-1)(7) = 5,$$

and $$N_y = \begin{vmatrix} 3 & -1 \\ 4 & 7 \end{vmatrix} = 3(7) - (-1)(4) = 25.$$

Then $$x = \frac{N_x}{D} = \frac{5}{10} = \frac{1}{2} \text{ and } y = \frac{N_y}{D} = \frac{25}{10} = \frac{5}{2}.$$

(2) Determinants of Order Three

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of 9 elements arranged in three rows and three columns, is called a determinant of order three. Its value is

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

This may be written as

$$a_1(b_2c_3 - b_2c_3) - b_1(a_2c_3 - b_3c_2) + c_1(a_2b_3 - a_3a_2)$$

or $$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

$$\therefore \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

The solution of the system of consistent and independent equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

is given by

$$x = \frac{N_x}{D} = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{N_y}{D} = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

and

$$z = \frac{N_z}{D} = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Example (5.7)

Solve, using determinants: $\begin{cases} x + 3y + 2z = -13 \\ 2x - 6y + 3z = 32 \\ 3x - 4y - z = 12. \end{cases}$

Solution:

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 2 & -6 & 3 \\ 3 & -4 & -1 \end{vmatrix} = 1(6+12) - 3(-2-9) + 2(-8+18)$$

$$= 18 + 33 + 20 = 71,$$

$$N_x = \begin{vmatrix} -13 & 3 & 2 \\ 32 & -6 & 3 \\ 12 & -4 & -1 \end{vmatrix} = -13(6+12) - 3(-32-36) + 2(-128+72)$$

$$= -234 + 204 - 112 = -142,$$

$$N_y = \begin{vmatrix} 1 & -13 & 2 \\ 2 & 32 & 3 \\ 3 & 12 & -1 \end{vmatrix} = 1(-32-36)-(-13)(-2-9)+2(24-96)$$

$$= -68 - 143 - 144 = -355,$$

$$N_z = \begin{vmatrix} 1 & 3 & -13 \\ 2 & -6 & 32 \\ 3 & -4 & 12 \end{vmatrix} = 1(-72+128)-3(24-96)+(-13)(-8+18)$$

$$= 56 + 216 - 130 = 142.$$

Then $x = \dfrac{N_x}{D} = -\dfrac{142}{71} = -2$, $y = \dfrac{N_y}{D} = \dfrac{-355}{71} = -5$, and $z = \dfrac{N_z}{D} = \dfrac{142}{71} = 2$.

5.5 The Inverse matrix

(1) Definition (The Inverse Matrix)

The inverse of a given square matrix is a matrix such that when multiplied on the right or on the left by the given matrix yields the unit matrix. We denote the inverse of matrix A by A^{-1}. Then we have

$$A A^{-1} = A^{-1} A = I,$$

where I is the unit matrix.

(2) Definition (Singular and Non-singular Matrix)

A square matrix is non-singular if the determinant of A is different from zero, otherwise it is called a singular matrix.

(3) Theorem

Every non-singular matrix has an inverse.

Proof: Omit.

If we have a non-singular matrix of order n, then

$$A^{-1} = \frac{Adj\ A}{\det A}$$

where $Adj\ A$ is the adjoint of matrix A:

$$Adj\ A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \cdots & \alpha_{n2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \cdots & \alpha_{nn} \end{bmatrix}$$

and α_{ij} are the cofactors (signed minors) of the corresponding entries a_{ij} $(i, j = 1, 2, \ldots, n)$.

Example (5.8)

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Solution:

Since
$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ -2 & -4 & -5 \\ 3 & 5 & 6 \end{vmatrix}$$
$$= 1(-24+25) - 2(-12+15) + 3(-10+12)$$
$$= 1 - 6 + 6 = 1 \neq 0$$

$\therefore$ A is non-singular.

The cofactors are

$$\alpha_{11} = +|A_{11}| = \begin{vmatrix} -4 & -5 \\ 5 & 6 \end{vmatrix} = (-4)(6) - (-5)(5) = 1$$

$$\alpha_{12} = -|A_{12}| = -\begin{vmatrix} -2 & -5 \\ 3 & 6 \end{vmatrix} = -[(-2)(6) - (-5)(3)] = -3$$

$$\alpha_{13} = +|A_{13}| = \begin{vmatrix} -2 & -4 \\ 3 & 5 \end{vmatrix} = (-2)(5) - (-4)(3) = 2$$

$$\alpha_{21} = -|A_{21}| = -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -[(2)(6) - (3)(5)] = 3$$

$$\alpha_{22} = +|A_{22}| = \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} = (1)(6) - (3)(3) = -3$$

$$\alpha_{23} = -|A_{23}| = -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -[(1)(5) - (2)(3)] = 1$$

$$\alpha_{31} = +|A_{31}| = \begin{vmatrix} 2 & 3 \\ -4 & -5 \end{vmatrix} = (2)(-5) - (3)(-4) = 2$$

$$\alpha_{32} = -|A_{32}| = -\begin{vmatrix} 1 & 3 \\ -2 & -5 \end{vmatrix} = -[(1)(-5) - (3)(-2)] = -1$$

$$\alpha_{33} = +|A_{33}| = \begin{vmatrix} 1 & 2 \\ -2 & -4 \end{vmatrix} = (1)(-4) - (2)(-2) = 0$$

Therefore the adjoint matrix is

$$Adj\ A=\begin{bmatrix}1 & 3 & 2\\ -3 & -3 & -1\\ 2 & 1 & 0\end{bmatrix}$$

and hence $$A^{-1}=\frac{Adj\ A}{\det A}=\begin{bmatrix}1 & 3 & 2\\ -3 & -3 & -1\\ 2 & 1 & 0\end{bmatrix}.$$

Problem Set (5.1)

5.1.1 Let $A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$ and $B=\begin{pmatrix}1 & 0 & 1\\ 2 & 5 & 2\end{pmatrix}$.

Find (i) $3A+4B$, (ii) $5A-2B$.

5.1.2 Let $A=\begin{pmatrix}1 & -1\\ 1 & 0\\ 3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}1 & -2 & -5\\ 3 & 4 & 0\end{pmatrix}$.

Find (i) AB, (ii) BA.

5.1.3 Let $A=\begin{pmatrix}1 & 2 & 0\\ 3 & -4 & 5\\ 0 & -1 & 2\end{pmatrix}$ and let $f(x)=x^2-4x+3$. Find $f(B)$.

5.1.4 Let $A=\begin{pmatrix}1 & 2 & 0\\ 3 & -1 & 4\end{pmatrix}$. Find (i) AA^T, (ii) A^TA.

5.1.5 Let $A=\begin{pmatrix}2 & 2\\ 2 & -1\end{pmatrix}$. Find (i) A^2, (ii) A^3, and (iii) if $f(x)=x^2+5x-4$, find $f(A)$.

5.1.6 Let $A=\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 2 & 1 & 3\end{pmatrix}$ and $B=\begin{pmatrix}2 & 0 & 1\\ 1 & -1 & 0\\ 0 & 2 & 1\end{pmatrix}$. Find (i) $(A+B)^T$, (ii) $(AB)^T$, (iii) $(A-B)^T(A+B)^T$, and (iv) $(A^2-B^2)^T$.

Problem Set (5.2)

5.2.1 Evaluate the following determinants:

$$(a)\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}, \quad (b)\begin{vmatrix} 5 & -2 \\ 3 & 1 \end{vmatrix}, \quad (c)\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix},$$

$$(d)\begin{vmatrix} 2 & 0 \\ -5 & 1 \end{vmatrix}, \quad (e)\begin{vmatrix} 2 & -10 \\ 3 & -15 \end{vmatrix}, \quad (f)\begin{vmatrix} 2 & -2 & -1 \\ 3 & 1 & -1 \\ 4 & 3 & 5 \end{vmatrix},$$

$$(g)\begin{vmatrix} 2 & 5 & 0 \\ 0 & 3 & 4 \\ -5 & 3 & 6 \end{vmatrix}, \quad (h)\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & 4 \\ 4 & 2 & 3 \end{vmatrix}, \quad (i)\begin{vmatrix} 4 & -3 & 2 \\ 5 & 9 & -7 \\ 4 & -1 & 7 \end{vmatrix}.$$

5.2.2 Solve, using determinants:

(a) $x+2y=-4$
$5x+3y=1$

(b) $ax-2by=c$
$2ax-3by=4i$

(c) $3x+2z=8-5y$
$3y+2x=z+1$
$3z-1=x-2y$

(d) $2x-5y+2z=7$
$x+2y-4z=3$
$3x-4y-6z=6$

(e) $2z+3=y+3x$
$x-3z=2y+1$
$3y+z=2-2x$

(f) $2x-3y+2z=6$
$x+8y+3z=-31$
$3x=2y+z=-5$.

5.2.3 Solve, using adjoint matrix:

$$\begin{aligned} x+2y+2z&=4 \\ 3x-y+4z&=25 \\ 3x+2y-z&=-4. \end{aligned}$$

Problem Set (5.3)

5.3.1 Find the inverse of the matrix:

$$(i)\ A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad (ii)\ B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix},$$

$$(iii)\ C=\begin{pmatrix} 2 & -3 & 4 \\ 1 & 2 & -3 \\ -1 & -2 & 5 \end{pmatrix}, \quad (iv)\ D=\begin{pmatrix} 4 & -1 & -2 \\ 0 & 2 & -3 \\ 5 & 2 & 1 \end{pmatrix}.$$

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