CHAPTER 6

INFLUENCE LINES FOR BEAMS, FRAMES, AND GRIDS

6.1 Introduction

The effect of live loads which can have different positions on a structure can be conveniently analyzed and succintly described in graphical form by the use of influence lines. An influence line shows the value of any action due to a unit point load moving across the structure. For example, the influence line for the bending moment at a section of a continuous beam shows the variation in the bending moment at this section as a unit transverse load traverses the beam.

In this chapter we deal with the methods of obtaining influence lines for statically indeterminate structures but, by way of introduction and review influence lines for statically determinate structures will be first briefly discussed.

6.2 Concept and Application of Influence Lines

A transverse concentrated load at a general position on a member of a structure causes various actions. These actions which may be a bending moment, shearing force, thrust, or displacement at a section, or a reaction at a support, vary as the load moves across the structure.

If the values of any action A are plotted as ordinates at all the points of application of a unit transverse load, we obtain the influence line of the action A. In this chapter we use η to represent the influence ordinate-which may also be referred to as influence coefficient-of any action due to a unit moving concentrated load acting at right angles to the member over which the load is moving. The effects of other types of loading will be distinguished by appropriate subscripts. Our sign convention is to plot positive influence ordinates in the same direction as the applied concentrated load. Thus influence lines for gravity loads on horizontal members are drawn positive downwards.

Let us now illustrate the use of influence lines in analysis. The value of any action A due to a system of concentrated loads $P_1, P_2, \ldots, P_n$ (Fig. 6-1a)

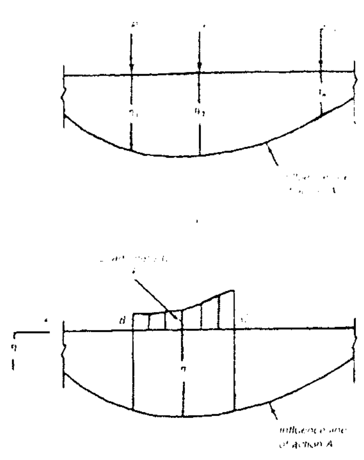


Fig. 6-1 Determination of the value of an action due to loading using the influence line

can be obtained from the influence ordinates by

$$A = \eta_1 P_1 + \eta_2 P_2 + \ldots\ldots + \eta_n P_n \quad (6.1)$$

or

$$A = \sum_{i=1}^{n} \eta_i P_i \quad (6.1a)$$

The value of the action A due to a distributed transverse load of intensity p over a length BC (Fig. 6.1b) is

$$A = \int_B^C \eta p dx \quad (6\text{-}2)$$

For a uniform load of intensity q,

$$A = q \int_B^C \eta dx \quad (6\text{-}3)$$

The value of the integral in this equation is the area under the influence line between B and C.

The knowledge of the shape of an influence line indicates which part or parts of a structure should be loaded to obtain maximum effects. In Fig.6-4, influence lines are sketched for a plane frame, and in Fig 6-5 for a grid. The ordinates plotted on the column EB in Fig.6-4 represent the value of the action considered due to a horizontal load on the column. As always; the value is positive if the load is applied in the direction of the positive ordinate. The ordinates of the influence lines for the grid are vertical to represent the effect of a unit vertical load, as shown in the pictorial view in Fig 6-5.

We can see that, for instance, in the case of shear at section n in the frame of Fig.6-4, a maximum negative value occurs when a distributed load covers Bn as well as the span CD, without a load on the remainder of the frame. Likewise, the bending moment at n_3 in Fig 6-5 is maximum positive when loads cover the members CD and the central part of GI, without a load on AB, or EF.

6.3 Muller-Breslau's Principle

One of the most effective methods of obtaining influence lines is by the use of Muller-Breslau's principle, which states that the ordinates of the influence line for any action in a structure are equal to those of the deflection curve obtained by releasing the restraint corresponding to this action and introducing a corresponding unit displacement in the remaining structure. The principle is applicable to any structure, statically determinate or indeterminate, and can be easily proved, using Betti's law.

Consider a loaded beam in equilibrium, as in Fig. 6.2a. Remove the support B and replace its effect by the corresponding reaction R_B, as shown in Fig. 6.2b. If the structure is now subjected to a downward load F at B such that the deflection at B equals unity, the beam will assume the deflected form in Fig. 6.2c. Because the original structure is statically determinate, the release of one restraining force turns the structure into a mechanism, and therefore the force F required to produce the displacements in Fig. 6.2c is zero. However, the release of one restraining force in a statically indeterminate structure leaves a stable structure so that the value of the force F is generally not equal to zero.

Applying Betti's Law to the two system of forces in Fig 6-2b and c, we write

$$\eta_1 P_1 + \eta_2 P_2 + --- + \eta_n P_n - 1 \times R_B = F \times 0$$

This equation expresses the fact that the external virtual work done by the system of forces in Fig. 6-2b during the displacement by the system in Fig 6-2c is the same as the external virtual work done by the system in Fig. 6-2c during the displacement by the system in Fig 6-2b. This latter quantity must be zero because no deflection occurs at B in Fig. 6.2b.

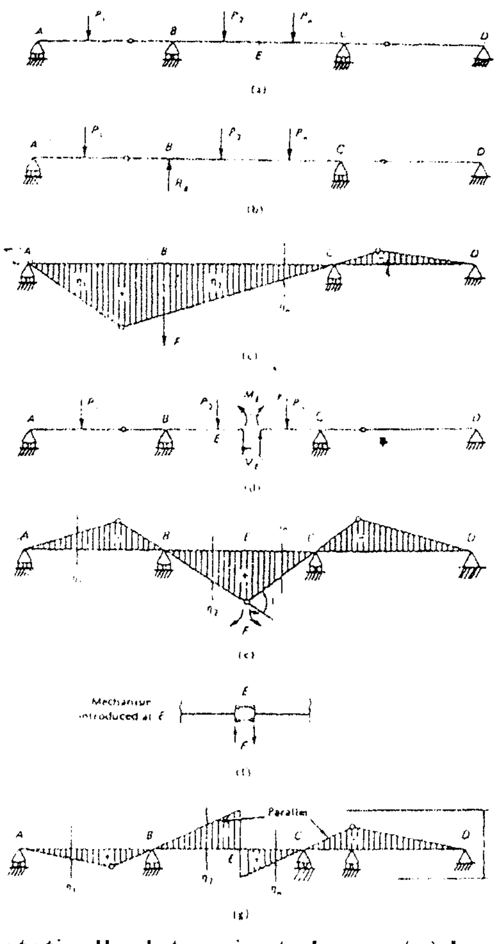


Fig 6-2. Influence line for a statically determinate beam. (a) Loaded beam in equilibrium. (b) Support B replaced by R_B (c) Influence line for R_B (d) Equilibrium maintained by forces M_E and V_E. (e) Influence line for M_E (g) Influence line for V_B

The preceding equation can be written

$$R_B = \sum_{i=1}^{n} \eta_i P_i$$

Comparing this equation with Eq. 6-1a, we see that the deflection line in Fig. 6-2c is the influence line of the reaction R_B. This shows that the influence line of the reaction R_B can be obtained by releasing its effect, that is removing the support B, and introducing a unit displacement at B in the downward direction, that is opposite to the positive direction of the reaction.

Using simple statics, we can readily check that the deflection ordinate at any point in Fig. 6-2c is, in fact, equal to the reaction R_B if a unit load is applied at this point in the beam of Fig. 6-2a.

Let us now use Muller-Breslau's principle in the case of the influence line of the bending moment at any section E. We introduce a hinge at E, thus releasing the bending moment at this section. We then apply two equal and opposite couples F to produce a unit relative rotation of the beam ends at E (Fig. 6-2e). In order to prove that the deflection line in this case is the influence line of the bending moment at E, cut the beam in Fig. 6-2a at section E and introduce two pairs of equal and opposite forces M_E and V_E to maintain the equilibrium (Fig. 6-2d). Applying Betti's law to the systems in Figs. 6-2d and 6-2e, we can write

$$\eta_1 P_1 + \eta_2 P_2 + \cdots + \eta_n P_n - 1 \times M_E = F \times 0$$

or

$$M_E = \sum_{i=1}^{n} \eta_i P_i$$

This demonstrates that the deflection line in Fig. 6.2e is the influence line for the bending moment at E.

The influence line for shear at section E can be obtained by introducing a unit relative translation without relative rotation of the two beam ends at E (Fig. 6.2g). This is achieved by introducing at E a fictitious mechanism such as that shown in Fig. 6.2f and then applying two equal and opposite vertical forces F. With this mechanism the two ends at E remain parallel as shown in Fig 6-2g. Applying Betti's Law to the systems in Figs. 6.2d and 6.2g, we can write

$$\eta_1 P_1 + \eta_2 P_2 + \cdots + \eta_n P_n - 1 \times V_E = F \times 0$$

or

$$V_E = \sum_{i=1}^{n} \eta_i P_i$$

which shows that the deflection line in Fig. 6-2f is the influence line for the shear E.

All the influence lines considered so far are composed of straight-line segments. This is the case for any influence line in any statically determinate structure. Thus, one computed ordinate and known shape of the influence line are sufficient to draw it. This ordinate may be calculated from considerations of statics, or from the geometry of the influence line.

All influence lines for statically indeterminate structures are composed of curves, and therefore several ordinates must be computed. In Fig. 6.3, Muller-Breslau's principle is used to obtain the general shape of the influence lines for a reaction, bending moment, and shear at a section in a continuous beam. Sketches of influence lines for several actions in a plane frame and in a grid are deduced by Muller-Breslau's principle in Figs. 6.4 and 6.5.

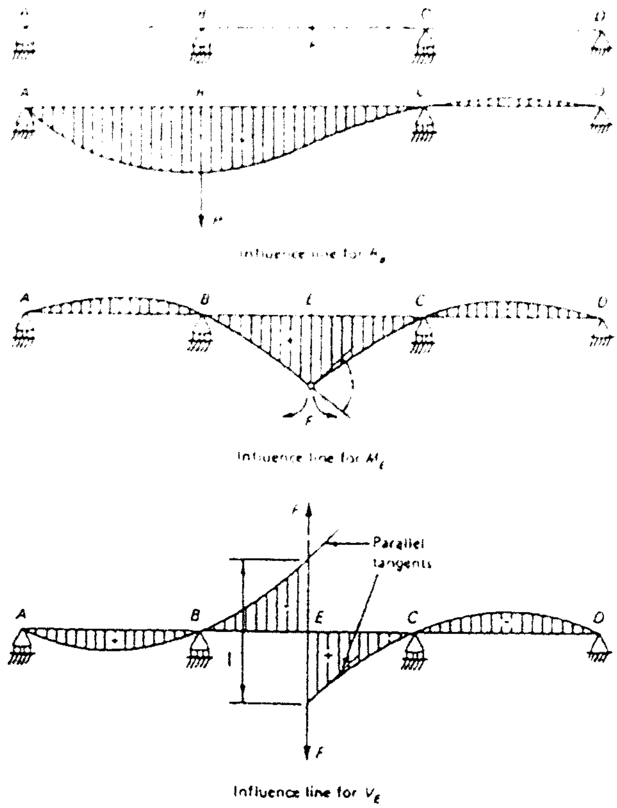


Fig. 6-3 Influence lines for a continuous beam.

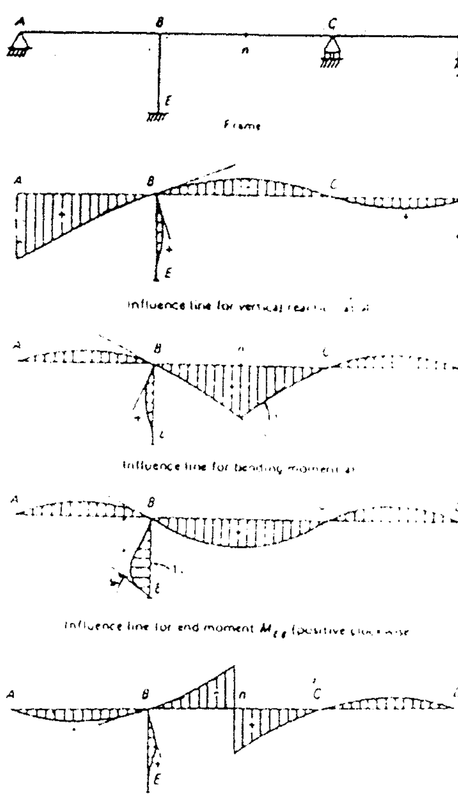
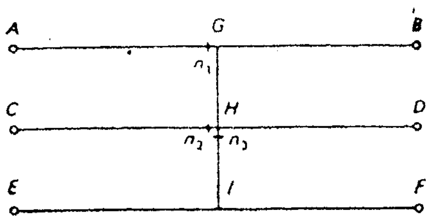


Fig. 6-4 Shape of influence lines for a plane frame using Muller-Breslaru's principle.

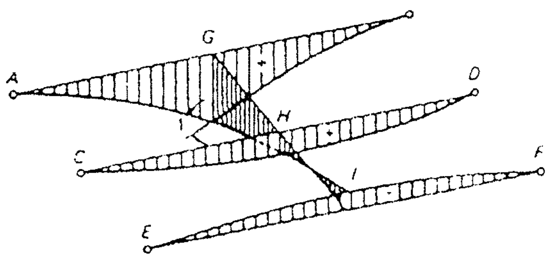
6.3.1 Procedure for Obtaining Influence Lines

The steps followed in Sec. 6.3 to obtain the influence line for any action can be summarized as follows.

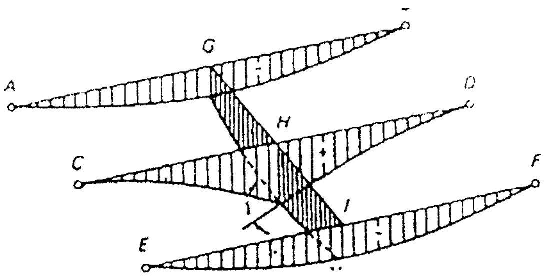
1. The structure is released by removal of the restraint corresponding to the action considered. The degree of indeterminacy of the released structure compared with the original structure is reduced by one. It follows that, if the original structure is statically determinate, the released structure is a mechanism.



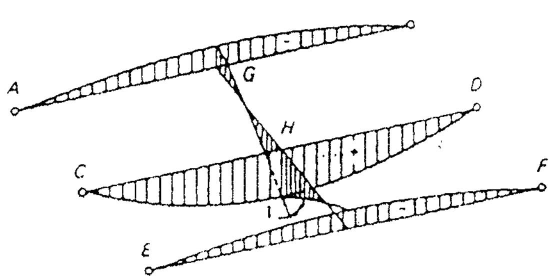
Ends A,C,E,B,D and F are simply supported



Influence line for bending moment at n_1



Influence line for bending moment at n_2



Influence line for bending moment at n

Fig 6-5 Shape of influence lines for a grid using Muller-Breslau's principle

2. Introduce a unit displacement in the released structure in a direction opposite to the positive direction of the action. This is achieved by applying a force (or a pair of equal and opposite forces) corresponding to the action.

3. The ordinates of the deflection line thus obtained are the influence ordinates of the action. The ordinates of the influence line are positive if they are in the same direction as the external applied load.

6-4 Correction for Indirect Loading

In some cases loads are not applied directly to the structure for which the influence lines are desired, but through smaller beams assumed to be simply supported on the main structure. For instance, the main girder in Fig. 6-6a supports cross-girders at nodes A, 1, 2, 3, and B, and these in turn

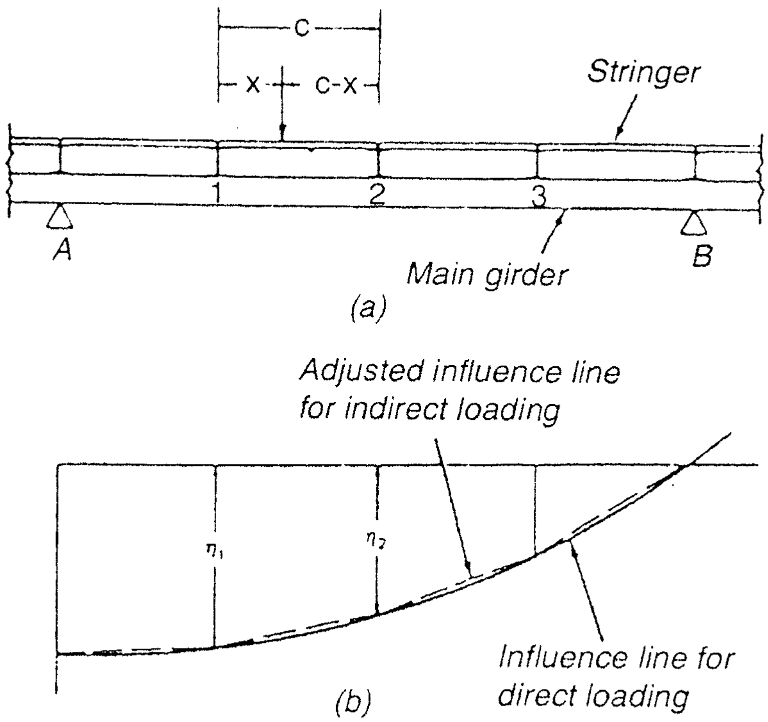


Fig 6.6 Correction of influence lines for indirect loading, (a) Indirect loading on main girder. (b) Influence line for any action *A* in main girder.

carry stringers to which the live load is applied. Let the solid curve in Fig. 6-6b represent the influence line of any action A, drawn on the assumption that the unit load is applied directly to the beam. But the unit load can be transmitted to the main girder at the cross-girders only, and we have to correct the influence line accordingly.

A unit load applied at an arbitrary point between nodes 1 and 2 is transmitted to the main girder as two concentrated loads equal to (c-x)/c and x/c at 1 and 2 respectively. The value of the action A due to these two loads is

$$A = \frac{x}{c}\eta_1 + \frac{c-x}{c}\eta_2 \qquad (6\text{-}4)$$

where c is the panel length and x is the distance indicated in Fig. 6-6a. This is the equation of the straight line between points 1 and 2 shown dotted in Fig. 6-6b. Thus the corrected influence line is composed of straight segments between the node points.

In pin-connected trusses, all the loads are assumed to act at the joints; thus, influence lines for trusses are composed of straight segments between the joints.

6.5 Influence Lines for a Beam with Fixed Ends

Let us now use Muller-Breslau's principle to find the influence lines for the end-moments of a beam with fixed ends. From these, by equations of statics, influence lines for reaction, shear, and bending moments at any section can be determined. We use-as in previous chapters-the convention that a clockwise end moment is positive.

To find the influence line for the end-moment M_{AB} in the beam in Fig. 6.7a, we introduce a hinge at A and apply there an anticlockwise moment to produce a unit angular rotation of the end A (Fig. 6.7b). This moment must be equal in magnitude to the end-rotational stiffness S_{AB}. The corresponding end-moment at B is $t = C_{AB} S_{AB}$, where S_{AB}, C_{AB} and t are the end-rotational stiffness, the carryover factor and the carryover moment respectively. The deflection line corresponding to the bending-moment, diagram in Fig. 6-7c is the required influence line.

When the beam has a constant flexural rigidity EI and length l, the end moments at A and B are respectively-4EI/ l and --2EI/l. These values can be substituted in the expression for the deflection y in a prismatic member AB due to clockwise end-moments M_{AB} and M_{BA}.

$$y = \frac{l^2}{6EI}[M_{AB}(2\varepsilon - 3\varepsilon^2 + \varepsilon^3) - M_{BA}(\varepsilon - \varepsilon^3)] \qquad (6\text{-}5)$$

where $\varepsilon = x / l$, x is the distance from the left-hand end A, and l is the length of the member. Equation 6-5 can be easily proved by the method of elastic weights. The values of y due to unit end moments are given in Appendix I.

Superposition of the deflections caused by an end-moment of $-4EI/l$ at A (with zero moment at B) and of the deflections caused by an end-moment

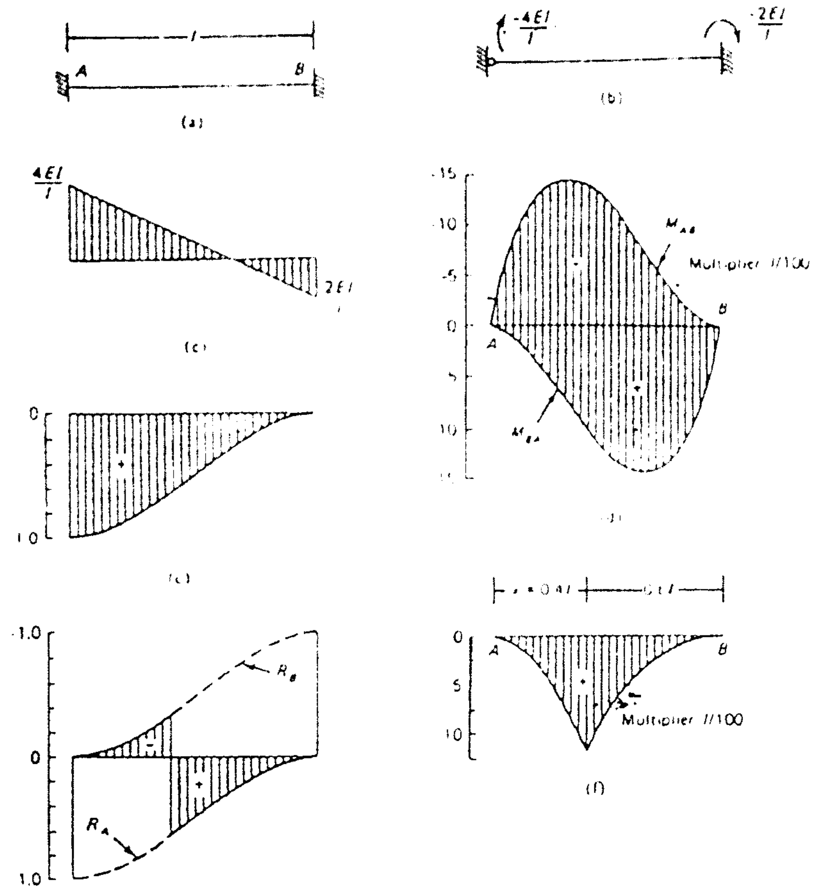


Fig. 6-7. Influence lines for a prismatic beam with fixed ends. (a) Beam. (b) End - moments corresponding to a unit angular rotation at end A. (c) Bending moment diagram for the beam in part (b). (d) Influence lines for end-moments. (e) Influence line for R_A. (f) Influence line for $M_{(x=0.4l)}$ (g) Influence line for $V_{(x=0.4l)}$

of $-2EI/l$ at B (with zero moment at A) gives the required influence line. This is conveniently done in Table 6-1.

Because the beam is symmetrical, the influence ordinates of the end-moment M_{BA} can be obtained from those for M_{AB} by reversing the sign and the order (Table 6.2).

The influence lines of the two end-moments are plotted in Fig. 6.7d.

The reaction R_A may be expressed as

$$R_A = R_{As} - \frac{M_{AB} + M_{BA}}{l}$$

Table 6-1. Calculation of Ordinates of the Influence Line for the End-Moment M_{AB}

Distance from left-hand end	0.1*l*	0.2*l*	0.3*l*	0.4*l*	0.5*l*	0.6*l*	0.7*l*	0.8*l*	0.9*l*	Multiplier
Deflection due to end moment at A of $-4EI/l$	-0.114	-0.192	-0.238	-0.256	-0.250	-0.224	-0.182	-0.128	-0.066	*l*
Deflection due to end moment at B of $-2EI/l$	0.033	0.064	0.091	0.112	0.125	0.128	0.119	0.096	0.057	*l*
Influence ordinate for M_{AB}	-0.081	-0.128	-0.147	-0.144	-0.125	-0.096	-0.063	-0.032	-0.009	*l*

Table 6-2 Ordinates of the Influence Line for the End-Moment M_{BA}

Distance from left-hand end	0.1*l*	0.2 *l*	0.3 *l*	0.4 *l*	0.5 *l*	0.6 *l*	0.7 *l*	0.8 *l*	0.9 *l*	Multi-plier
Influence ordinates for M_{BA}	0.009	0.032	0.063	0.096	0.125	0.144	0.147	0.128	0.081	*l*

where R_{As} is the statically determinate reaction of the beam AB if simply supported. This equation is valid for any position of a unit moving load. We can therefore write,

$$\eta_{RA} = \eta_{RAs} - \frac{1}{l}(\eta_{MAB} + \eta_{MBA}) \qquad (6.6)$$

where η is the influence ordinate of the action indicated by the subscript. The influence line of R_{AS} is a straight line with ordinate 1 and A and zero at

Table 6-3 Ordinates of the Influence Line for R_A

Distance from left-hand end	0	0.1*l*	0.2 *l*	0.3 *l*	0.4 *l*	0.5 *l*	0.6 *l*	0.7 *l*	0.8 *l*	0.9 *l*	*l*
η_{RAS}	1.000	0.900	0.800	0.700	0.600	0.5	0.400	0.300	0.200	0.100	*0*
$-\eta_{MAB}/l$	0	0.081	0.128	0.147	0.144	0.125	0.096	0.063	0.032	0.009	*0*
$-\eta_{MBA}/l$	0	-0.009	-0.032	-0.063	-0.096	-0.125	-0.144	-0.147	-0.128	-0.081	*0*
Influence ordinate for R_A	1.000	0.972	0.896	0.784	0.648	0.500	0.352	0.216	0.104	0.028	*0*

B. The calculation of the influence line ordinates for the reaction R_A is performed in Table 6-3, and the influence line is plotted in Fig. 6-7e.

Similarly, the influence ordinate for the bending moment at any section distance x from the left-hand end is given by

$$\eta_{M} = \eta_{Ms} + \frac{(l-x)}{l}\eta_{MAB} - \frac{x}{l}\eta_{MBA} \qquad (6\text{-}7)$$

where η_M and η_{Ms} are the influence ordinates for the bending moment at the section for a beam with fixed ends and simply supported respectively. The ordinates η_M for a section x = 0.4*l* are calculated in Table 6-4 and Fig. 6-7f plots the relevant influence line.

Table 6-4 Ordinates of the Influence Line for $M_{(x = 0.4l)}$

Distance from left-hand end	0.1*l*	0.2 *l*	0.3 *l*	0.4 *l*	0.5 *l*	0.6 *l*	0.7 *l*	0.8 *l*	0.9 *l*	*Multi-plier*
η_{ms}	0.060	0.120	0.180	0.240	0.200	0.160	0.120	0.080	0.040	*l*
$0.6\eta_{MAB}$	-0.049	-0.077	-0.088	-0.086	-0.075	-0.058	-0.038	-0.019	-0.005	*l*
$-0.4\eta_{MBA}$	-0.004	-0.013	-0.025	-0.018	-0.050	-0.058	-0.059	-0.051	-0.032	*l*
Influence ordinate for $M_{(x=0.4l)}$	0.007	0.030	0.062	0.116	0.075	0.046	0.023	0.010	0.003	*l*

The influence ordinates η_V of the shear at any section can be calculated by the equation

$$\eta_v = \eta_{vs} - \frac{1}{l}(\eta_{MAB} + \eta_{MBA}) \tag{6.8}$$

where η_{Vs} is the influence ordinate for the shear at the same section in a simply supported beam. The influence line for shear at a section $x = 0.4l$ is shown in Fig. 6-7g. It can be seen that this influence line can be formed by parts of the influence lines for R_A and R_B

The influence lines for continuous prismatic beams with equal spans or with unequal spans in certain ratios are given in various references, and in most cases they need not be calculated. On the other hand, influence lines are often calculated in the design of bridges of variable I or with irregularly varying spans forming continuous beams, also of frames and grids.

6-6 Influence Lines for Plane Frames

In the preceding section we have seen that the influence lines for shear or bending moment at any section of a member can be determined from the influence lines for the end-moments by simple equations of statics. Thus influence lines for end-moments are of fundamental importance, and we shall now show how to use moment distribution to find the influence lines for the end-moments of continuous plane frames.

Let us assume that we want to find the influence line for the end-moment

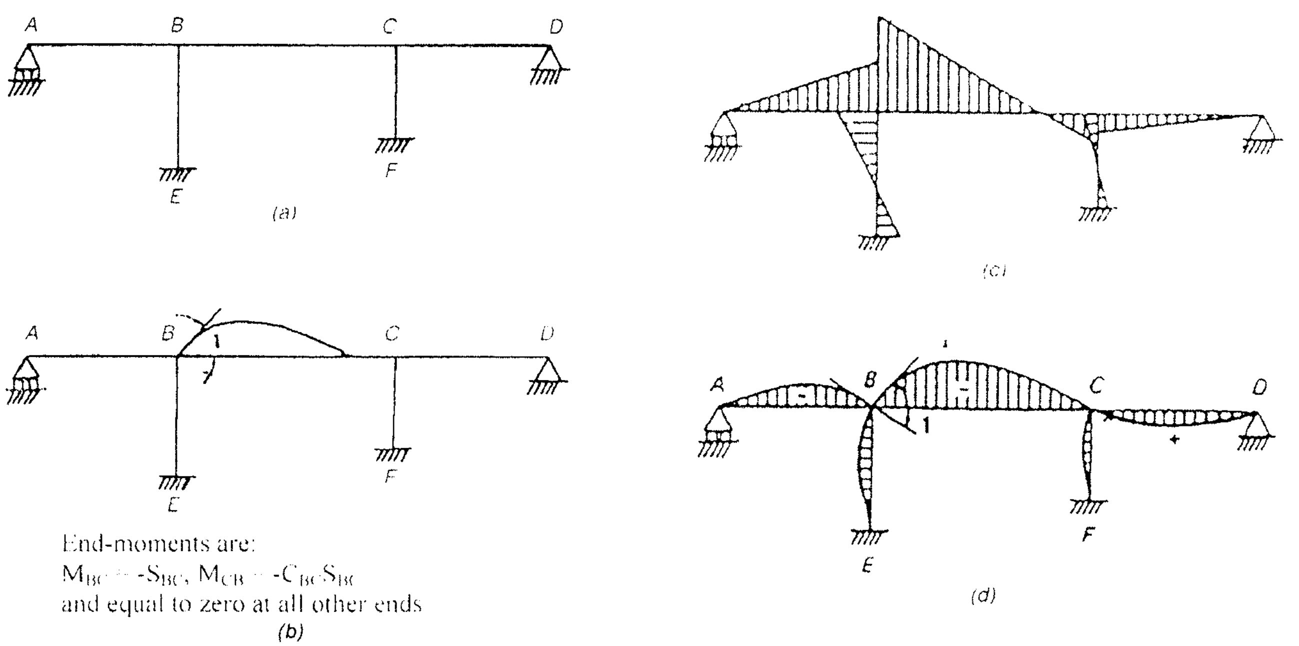


Fig. 6-8 Determination of influence line for end-moment in a plane frame, (a) Plane frame, (b) Unit angular rotation of end BC without other joint displacements, (c) Bending-moment diagram corresponding to the elastic line in part, (d) Influence line for the end-moment M_{BC}

M_{BC} in the frame of Fig. 6-8a. According to the Muller-Breslau principle, the influence ordinates are the ordinates of the deflected shape of the frame corresponding to a unit angular discontinuity at end BC. Assume that such a unit angular rotation is introduced at end BC without other displacements at the joints, as shown in Fig. 6-8b. The end-moments corresponding to this configuration are $-S_{BC}$ and $-t_{BC} = -C_{BC}S_{BC}$, where S_{BC} is the end rotational stiffness, t_{BC} the carryover moment, and C_{BC} the carryover factor from B to C.

We now allow joint rotations (and joint translations, if any) to take place and find the corresponding moments at the ends of the members by moment distribution in the usual way. The corresponding bending-moment diagram will be a straight line for each member (Fig 6-8c). The deflections, which are the influence line ordinates, are calculated by superposition of the deflections due to the end-moments as in the previous section.

For prismatic members, the values given in Appendix I may be used. For members of variable *I*, we can use the influence line ordinates of the moment at a fixed end of a member with the other end hinged. To obtain the deflection due to a unit couple applied at one end, these ordinates should be divided by the adjusted end-rotational stiffness at the fixed end while the other end is hinged.

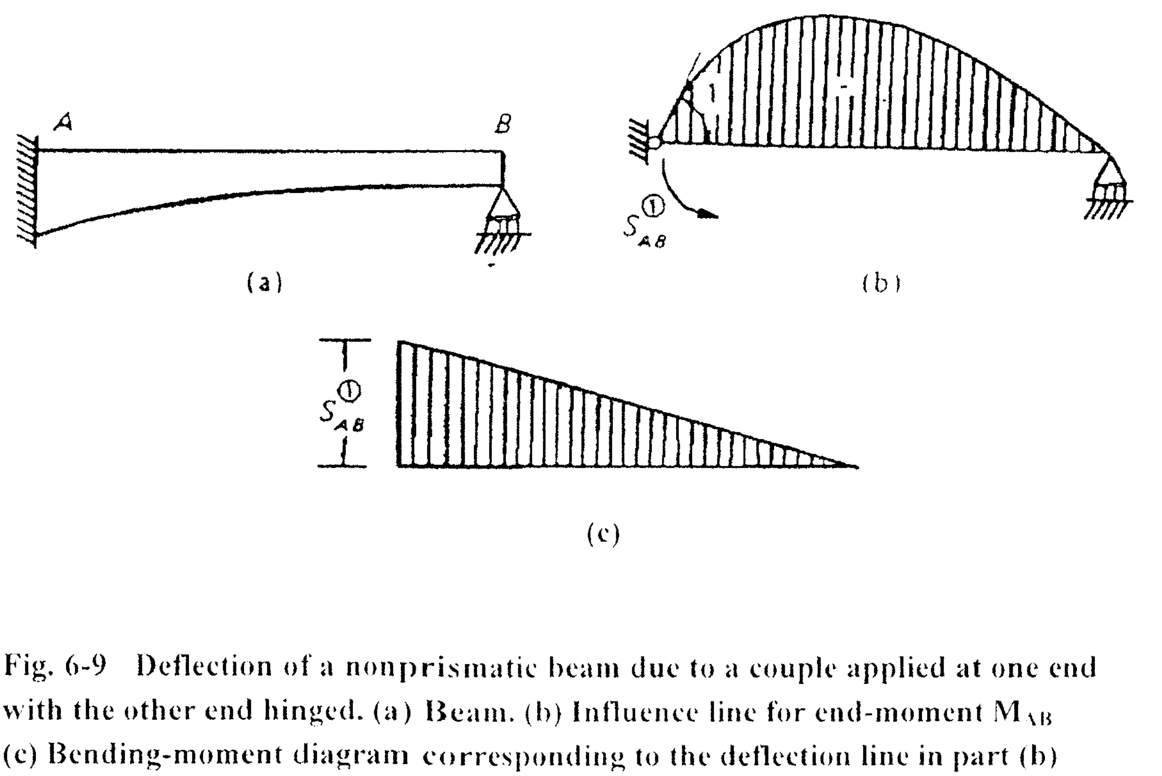
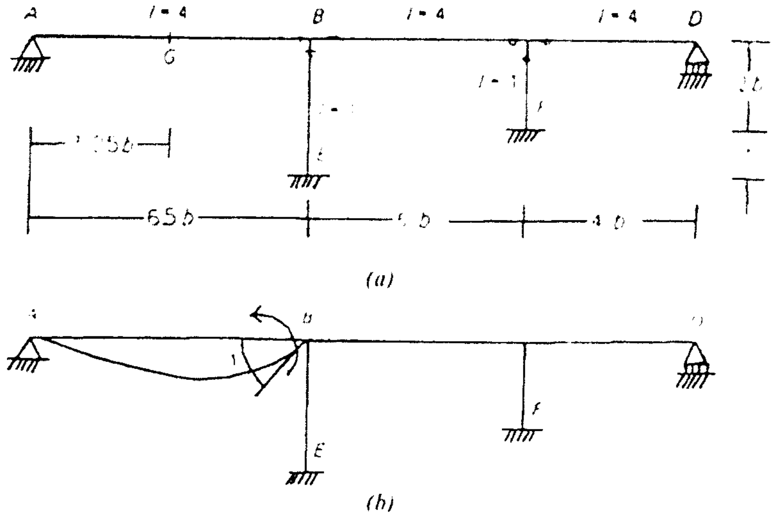


Fig. 6-9 Deflection of a nonprismatic beam due to a couple applied at one end with the other end hinged. (a) Beam. (b) Influence line for end-moment M_{AB} (c) Bending-moment diagram corresponding to the deflection line in part (b)

The shape of the influence line for the end moment M_{BC} for the frame considered is shown in Fig. 6-8d. The ordinates plotted on the columns BE and CF can be used to find the value of M_{BC} if a unit horizontal load is applied to either of the columns. The value will be positive if the load points toward the left. If however, a horizontal load on a column cannot occur; the influence ordinates on BE and CF need not be plotted.

Example 6.1 Obtain the influence line for the end-moment M_{BA} in the bridge frame in Fig 6-10a. Use this influence line to find the influence ordinate of the bending moment M_G at the center of AB and of the shear

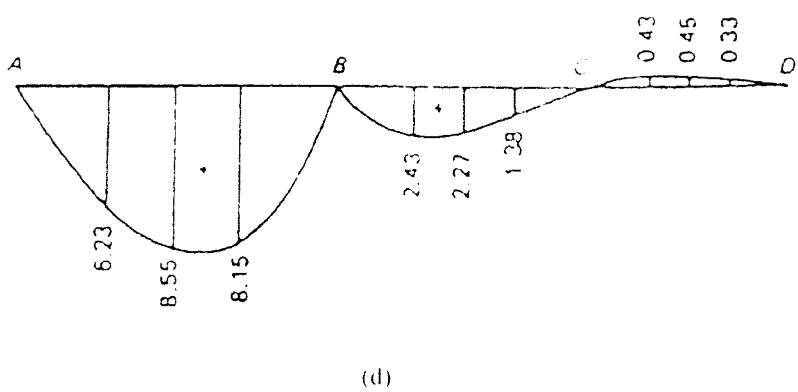


multiplier :
EI / (100 b)

End	BA	BE	BC		CB	CF	CD
	0.29	0.21	0.50		0.39	0.24	0.37
FEM's	- 185	0	0		0	0	0
	+ 53	+ 39	+ 93	→	+ 47		
			- 9	←	- 18	- 12	- 17
	+ 2	+ 2	+ 5	→	+ 3		
					- 1	- 1	- 1
Final end-moments	- 130	+ 14	+ 89		+ 31	- 13	- 18

(c)

multiplier :
b / 10



(d)

Fig. 6-10 Influence line for an end-moment in Example 6-1. (a) Frame properties.(b) Unit angular rotation introduced at end BA. (c) Moment distribution. (d) Influence line of end-moment M_{BA}

V_n at a point n just to the left of B. The relative values of I are shown in the figure.

A unit rotation in an anticlockwise direction is introduced at end B of BA, as shown in Fig. 6-10b. The corresponding end-moments are $M_{BA} = -3(EI/l)_{BA} = -1.85EI/b$ and zero for all the other ends. These values are the initial FEM's for which a moment distribution is carried out in Fig. 6-10c. The deflections of members AB, BC, and CD due to the final end-moments are calculated in Table 6-5 at $0.3l$, $0.5l$, and $0.7l$ of each span

Table 6-5 . Ordinates of Influence Line for End-Moment M_{BA}(b/10)

Deflection due to end-moment at	Member AB			Member BC			Member CD		
	$0.3l$	$0.5l$	$0.7l$	$0.3l$	$0.5l$	$0.7l$	$0.3l$	$0.5l$	$0.7l$
Left-hand end	0	0	0	3.31	3.48	2.53	-0.43	-0.45	-0.33
Right-hand end	6.23	8.55	8.15	-0.88	1.21	-1.15	0	0	0
Influence ordinate	6.23	8.55	8.15	2.43	2.27	1.38	-0.43	-0.45	-0.33

by the use of the tabulated values in Appendix I. These deflections, which are the influence ordinates of the end-moment M_{BA} are plotted in Fig. 6-10d. As always, a positive sign indicates a clockwise end-moment.

Table 6.6 Ordinates of Influence Line for the Bending Moment M_G at G (b/10)

Influence coefficient	Member AB			Member BC			Member CD		
	$0.3l$	$0.5l$	$0.7l$	$0.3l$	$0.5l$	$0.7l$	$0.3l$	$0.5l$	$0.7l$
η_{ms}	9.75	16.25	9.75	0	0	0	0	0	0
$-\frac{1}{2}\eta_{MBA}$	- 3.12	-4.28	- 4.08	- 1.22	- 1.14	- 0.69	0.22	0.23	0.17
Influence ordinate	6.63	11.97	5.67	- 1.22	- 1.14	- 0.69	0.22	0.23	0.17

The ordinates of the influence lines for M_G and V_n are determined by superposition Eq. 6.7 and 6.8 respectively. The calculations are performed in Tables 6.6 and 6.7, and the influence lines are plotted in Figs. 6.11a and b.

Table 6.7 Ordinates of Influence Line for Shear V_n

Influence coefficient	Member AB				Member BC			Member CD		
	$0.3l$	$0.5l$	$0.7l$	l	$0.3l$	$0.5l$	$0.7l$	$0.3l$	$0.5l$	$0.7l$
η_{VS}	-0.30	-0.50	-0.70	-1.00	0	0	0	0	0	0
$-\frac{1}{6.5b}(\eta_{MBA})$	-0.10	-0.13	-0.13	0	-0.04	-0.04	- 0.02	0.01	0.01	0.005
Influence ordinate	-0.40	-0.63	-0.83	-1.00	-0.04	- 0.04	-0.02	0.01	0.01	0.005

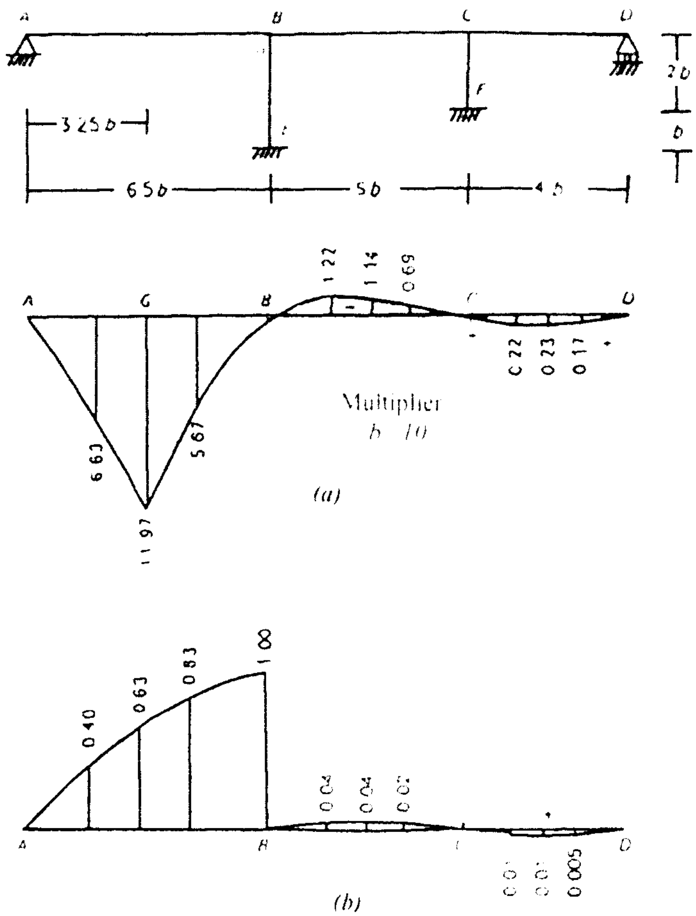


Fig 6-11, Influence line for bending moment and shear at a section of the frame in Example 6-1. (a) Influence line for M_G. (b) Influence line for V_B.

6.7 Influence Lines for Grids

The grid in Fig. 6-12 a represents the main and cross-girders of a bridge deck for which influence lines of bending moments at certain sections of the members are required. All joints are assumed to be rigid, capable of resisting bending and torsion.

For the analysis of this grid by the displacement method each of the internal joints has three unknown displacements: two rotations (vectors θ_x and θ_z) and a downward deflection δ. At each support two rotations, θ_x and θ_z are possible. One method of obtaining the influence lines is to carry out the analysis of the structure for a number of loading cases with a unit vertical load at various positions. Each loading case gives one ordinate for each influence line required. This method is satisfactory when a computer is used as little additional effort in programming is required in addition to that necessary for the dead-load analysis.

Another method of obtaining the influence lines, and one which required less computer time, is by the use of the Muller-Breslau principle, as discussed below.

We arrange the stiffness matrix of the grid in such a way that the elements corresponding to the vertical deflection occupy the first rows and the first

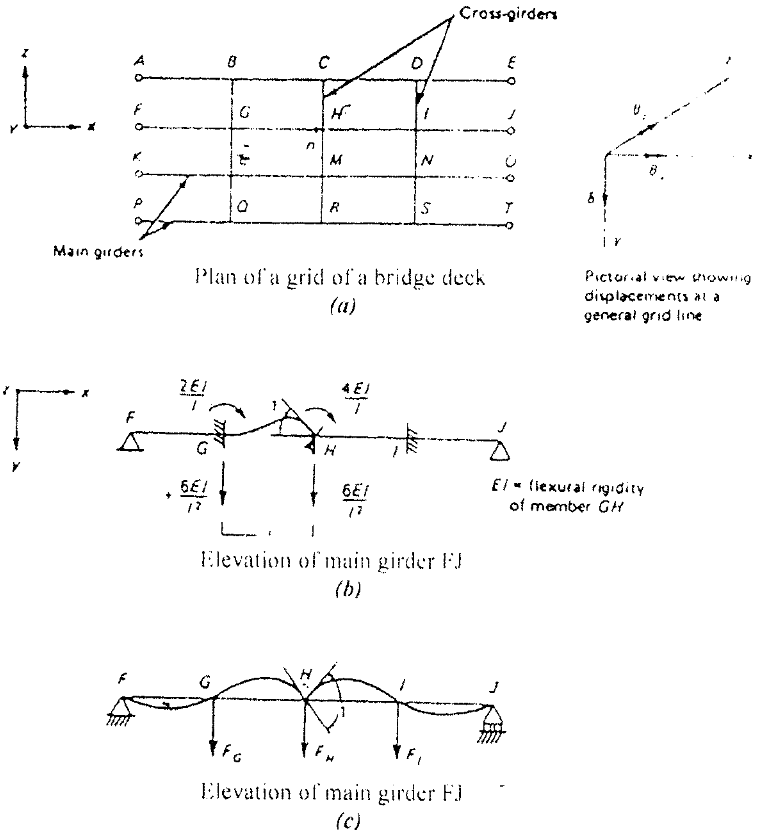


Fig.6-12. Determination of the influence line for the bending moment at section n of a grid. (a) Grid plan. (b) Restraining forces corresponding to a unit angular discontinuity at n without joint displacements. (c) Restraining forces corresponding to a unit angular discontinuity at n with the vertical displacements prevented at the joints.

columns. Let us then partition the matrix as follows

$$[S] = \left[\begin{array}{c|c} [A_{11}] & [A_{12}] \\ \hline [A_{21}] & [A_{22}] \end{array}\right] \tag{6-9}$$

For the grid in Fig. 6-12a, the order of $[S]$ is 52 x 52, and that of $[A_{11}]$ is 12 x 12. Assume that we require the influence line for the bending moment M_n at section n, just to the left of H. We induce a rotation at the end H of member. HG in the vertical plane, as shown in the elevation of girder FJ in Fig. 6-12b. The forces required to hold the structure in this configuration are two couples and two vertical forces shown in the figure, with no forces at all the other joints. If these forces are now released, the grid will deform maintaining a unit angular discontinuity at H between the members HG and HI. The vertical deflections of the grid are therefore the ordinates of the required influence line.

The deflections can be obtained by the equation

$$\begin{bmatrix} [A_{11}] & [A_{12}] \\ [A_{21}] & [A_{22}] \end{bmatrix} \begin{Bmatrix} \{D_1\} \\ \{D_2\} \end{Bmatrix} = -\begin{Bmatrix} \{F_1\} \\ \{F_2\} \end{Bmatrix} \qquad (6\text{-}10)$$

where the elements of $\{D_1\}$ and $\{D_2\}$ are respectively the vertical deflections and rotations at the joints. The elements of $\{F_1\}$ are all zero except for the elements corresponding to the vertical forces of $6EI/l^2$ at G and $-6EI/l^2$ at H. Similarly, the elements of $\{F_2\}$are all zero except for the elements corresponding to the couples in the vertical plane of $2EI/l$ at G and $4EI/l$ at H.

We want to solve Eq.6-10 in order to obtain $\{D_1\}$, whose elements are the ordinates of the influence line. Using Sec. A-7 of Appendix A, we can write

$$\{D_1\} = -\left[[A_{11}] - [A_{12}][A_{22}]^{-1}[A_{21}]\right]^{-1}\left\{\{F_1\} - [A_{12}][A_{22}]^{-1}\{F_2\}\right\} \qquad (6\text{-}11)$$

The matrix in the large square brackets in the above equation is the stiffness matrix of the grid corresponding to a system of vertical coordinates at the joints. The vector in the large braces in the same equation represents the forces along the vertical coordinates if the joints are allowed to rotate with the vertical displacement restrained.

Equation 6.11 gives the ordinates of the influence line at the nodes. If ordinates at points between the joints are required, the end-moments have to be determined first and the deflections from the straight lines joining the member ends are then determined by Eq. 6.5. The joint rotations $\{D_2\}$required to find the end-moments can be obtained from (see Sec. A-7 of Appendix A).

$$\{D_2\} = -[A_{22}]^{-1}\{\{F_2\} + [A_{21}]\{D_1\}\} \qquad (6\text{-}12)$$

If the torsional rigidity of the members is ignored, and the beams of the grid are equally spaced in each of the x and z directions, the stiffness matrix $[S]$ corresponding to a system of vertical coordinates at the joints can be easily obtained by the use of Appendix E. For the grid in Fig. 6-12a, this matrix is of the order 12 x 12. To find the ordinates of the influence line M_n, a unit rotation is introduced as in Fig. 6-12b, then, the joints F,G, H, I, and J ar allowed to rotate in the vertical plane without vertical displacement, and the corresponding restraining forces $\{F\}$ are determined (Fig.6.12c). This may be conveniently done by moment distribution with the girder FJ treated as a continuous beam.

The deflections at the joints are determined by $[S]\ \{D\} = -\{F\}$. For the grid in Fig. 6-12a, the elements of $\{D\}$ are the influence ordinates at the 12 intermediate joints, and the elements of $\{F\}$are all zero except for the three forces at joints G, H, and I: Ordinates between the joints, if required, can be calculated by Eq. 6-5. The end-moments needed for this equation can be determined from the vertical deflections, using the tabulated moments in Appendix E.

Example 6.2 Find the influence line for the bending moment at section n, just to the left of joint C, in the grid of Fig. 6-13a. The main girders are encastre. The relative moment of inertia is 4 for all main girders and 1 for the cross-girders. The ratio of the torsional rigidity GJ to the flexural rigidity EI is 1:4 for all members.

Figure 6.13b shows the coordinate system chosen, with three coordinates at each joint. The stiffness matrix for the grid is partitioned in the following manner:

$$[S] = \left[\begin{array}{c|c} [A_{11}] & [A_{12}] \\ (9 \times 9) & (9 \times 18) \\ \hline [A_{21}] & [A_{22}] \\ (18 \times 9) & (18 \times 18) \end{array}\right]$$

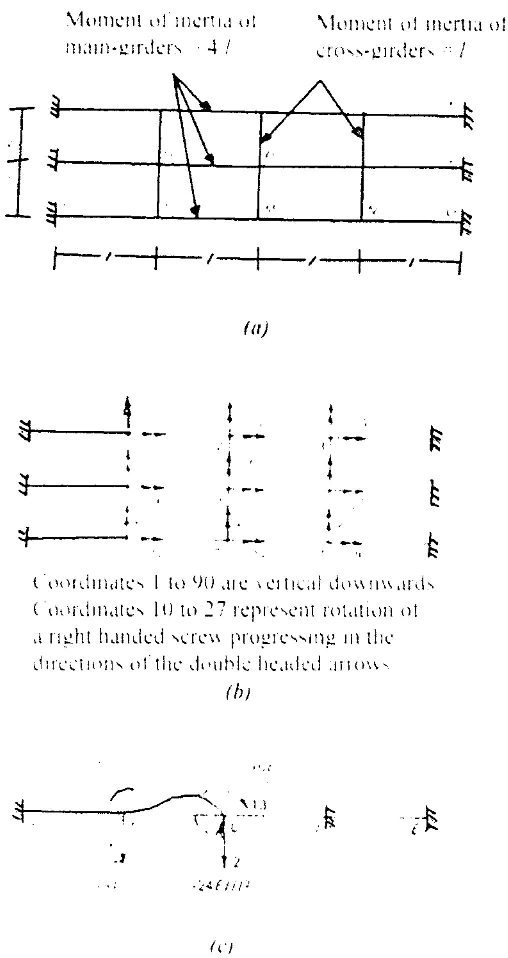


Fig. 6-13. Analysis of the grid of Example 6-2. (a) Grid plan. (b) Coordinate system. (c) Restraining forces corresponding to a unit angular discontinuity at n without joint displacements.

The order of each submatrix in this equation is indicated in brackets. The submatrices can be easily written with little calculation, and are given below. The matrices $[A_{11}]$ and $[A_{22}]$ are symmetrical, and $[A_{12}] = [A_{21}]^{T}$.

$[A_{11}] =$ $\frac{EI}{l^2}$ ×

	1	2	3	4	5	6	7	8	
1	192							symmetrical	
2	-48	192							
3		-48	192						
4	-96			288					
5		-96		-48	288				
6			-96		-48	288			
7				-96			192		
8	Element not			-96		-48	192		
9	shown are zero					-96		-48	192

$[A_{11}] = [A_{21}]^{T} =$

	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	24			24			24											
2		-24	24			24			24									
3				-24	24						24							
4	-24									24			24					
5			-24					-24				24			24			
6					-24					-24							24	
7	Element not							-24						-24			24	
8	shown are zero								-24					-24	-24			24
9											-24					-24		24

$[A_{22}] =$ $\frac{EI}{l}$ ×

	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
10	10																	
11		32.5																
12	-1		10															
13		8		32.5														
14			-1		10													
15				8		32.5												
16	4						18											
17		-0.5						33										
18			4				-1		18									
19			-0.5					8		33								
20					4				-1		18							
21						-0.5				8		33						
22							4						10					
23								-0.5						32.5				
24									4				-1		10			
25	Element not shown									-0.5				8		32.5		
26	are zero										4				-1		10	
27												-0.5				8		32.5

The restraining forces corresponding to a unit angular discontinuity at n without joint displacements are shown in Fig. 6-13c.

The submatrices $\{F_1\}$ and $\{F_2\}$ in Eq. 6-10 are (see the restraining forces in Fig. 6.13c):

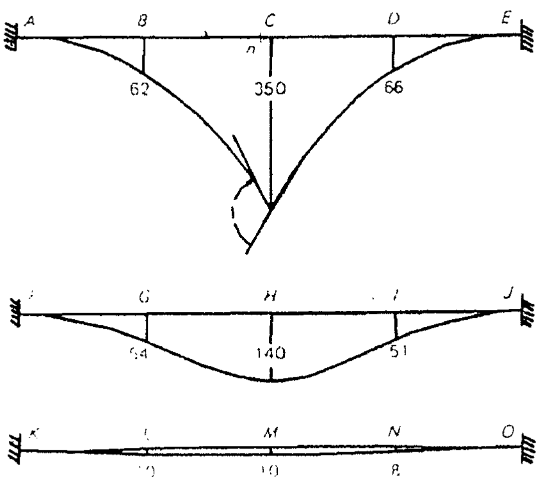
$$\{F_1\} = \frac{EI}{l^2}\{24, -24, 0, 0, 0, 0, 0, 0, 0\}$$

$$\{F_2\} = \frac{EI}{l}\{0, 8, 0, 16, \textit{and zero for all the other elements}\}$$

Substituting in Eq. 6-11, we obtain

$$\{D_1\} = 10^{-3}\, l\, \{62, 350, 66, 54, 140, 51, 10, 10, 8\}$$

The elements of $\{D_1\}$ are the ordinates of the influence line for the bending moment M_n at section n. These ordinates are plotted on an elevation of the main girders in Fig. 6.14.



Multiplier : $10^{3}\, l$

Fig. 6.14 Influence line for the bending moment M_n at section n of the grid in Example 6.2

The rotations $\{D_2\}$ obtained by Eq.6-12 are

$\{D_2\} = 10^{-3}\{-35, 134, -384, -488, -49, -140, -58, 106, -331, -10, -64, -101, -102, 10, -199, -2, -99, -9\}$

These rotations are used when influence ordinates between the joints are required.

Example 6.3 Neglecting the torsional rigidity of the girders, find for the interconnected bridge system of Fig. 6-15a the influence lines for the following actions:

(a) bending moment at the center of girder AB

(b) bending moment at the center of girder CD

(c) bending moment in the cross-girder at J

(d) reaction R_C at support C

The main girders are simply-supported and have a moment of inertia of 4I, where I is the moment of inertia of the cross-girder.

The stiffness matrix of the gird corresponding to vertical downward coordinates at N, J, K, and L (Fig. 6-15b) can be calculated from the values

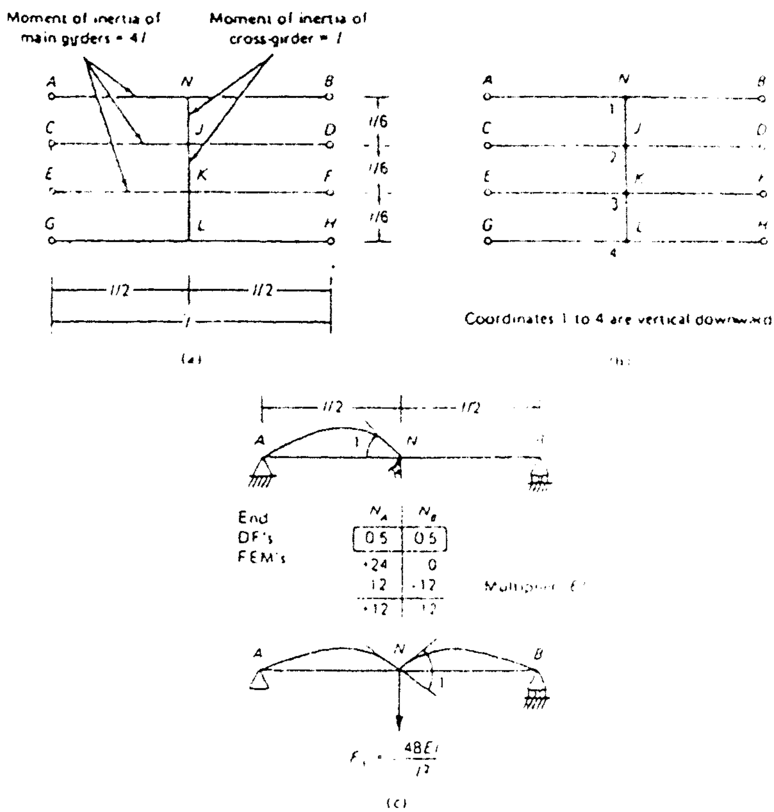


Fig. 6-15, Analysis of the torsionless grid in Example 6-3. (a) Grid plan.(b) Coordinate system. (c) Calculation of the restraining force F_1 corresponding to a unit angular discontinuity in main girder at N, with no displacements along the coordinate

tabulated in Appendix E. We obtain

$$[S] = \frac{EI}{l^3}\begin{bmatrix} 537.6 & & \text{symmetrical} & \\ -777.6 & 2265.6 & & \\ 518.4 & -1814.4 & 2265.6 & \\ -86.4 & 518.4 & -777.6 & 537.6 \end{bmatrix} \quad \text{(a)}$$

To find the influence line of the bending moment at the center of girder AB, we introduce a unit angular discontinuity just to the left (or right) of joint N (Fig. 6-15c) with the vertical joint displacements prevented. The end-moment M_{NA} corresponding to this configuration is

$$\frac{3E(4I)}{l/2} = 24\frac{EI}{l}$$

Moment distribution for the beam ANB is carried out in Fig. 6-15c, and the restraining force F_1 required to prevent the deflection at N is calculated

$$F = -48\frac{EI}{l^2}$$

It is obvious that no forces are required at the other three coordinates. Thus the matrix {F} is

$$\{F\} = \frac{EI}{l^2}\{-48, 0, 0, 0\}$$

The force F_1 can also be found by the use of the reactions tabulated in Appendix E, the procedure being as follows.

In order to reach the configuration of Fig. 6.16a we can proceed in two steps. First, we introduce a unit angular-discontinuity by allowing the left-hand end of the beam to lift by a distance $l/2$, as shown in Fig. 6.16b; no forces are involved. In the second step, the support A is brought back to its original level by a vertical downward force at A without a change in the angle between the ends of the members meeting at N. The value of the reaction at N due to a unit downward displacement of support A is (from Appendix E) $3E(4I)/(l/2)^3$

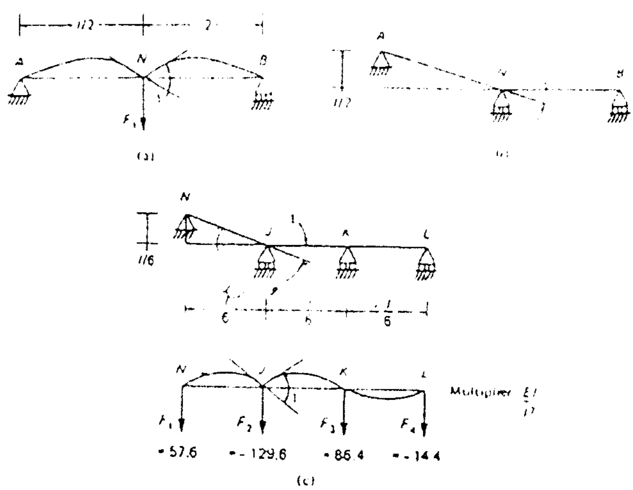


Fig. 6-16, Analysis of members of the torsionless grid in Example 6-3. (a) Unit angular discontinuity at N with the vertical deflection restrained by the force F_1, (b) Unit angular discontinuity introduced at N, without restraining forces. (c) Determination of the restraining forces corresponding to a unit angular discontinuity in the cross-girder at J by use of tabulated values in Appendix E.

upward. Therefore the value of the restraining force is $F_1 = -48(EI/l^2)$, which is the reaction at N corresponding to a downward displacement of $l/2$ at A.

Similarly, to find the influence line for the bending moment at the center of CD, we introduce a unit rotation at the end J of JC. The corresponding restraining forces are

$$\{F\} = \frac{EI}{l^2}\{0, -48, 0, 0\}$$

For the influence line for the bending moment in the cross-girder at section J, a unit angular discontinuity is introduced at this point, as shown in Fig. 6-16c, causing the lifting of end N by a distance $l/6$. The forces at N, J, K and L required to bring joint N to its original position, determined from Appendix E, are

$$\{F\} = \frac{EI}{l^2}\{57.6, -129.6, 86.4, -14.4\}$$

For the influence line of the reaction R_C, a unit downward displacement is introduced at C. The restraining force at J is taken from Appendix E, and the other restraining forces are zero. Therefore

$$\{F\} = \frac{EI}{l^3}\{0, -96, 0, 0\}$$

The influence ordinates for each of the four effects are summarized in the equation

$$[S][D] = -\frac{EI}{l^2}\begin{bmatrix} -48 & 0 & 57.6 & 0 \\ 0 & -48 & -129.6 & -\frac{96}{l} \\ 0 & 0 & 86.4 & 0 \\ 0 & 0 & -14.4 & 0 \end{bmatrix} \qquad \text{(b)}$$

where [S] is the stiffness matrix in Eq. (a). The solution of Eq. (b) gives

$$[D] = \begin{bmatrix} 0.194l & 0.082l & -0.038l & 0.164 \\ 0.082l & 0.097l & 0.055l & 0.194 \\ 0.010l & 0.062l & 0.006l & 0.124 \\ -0.034l & 0.010l & -0.024l & 0.020 \end{bmatrix}$$

The influence lines for the four actions are plotted in Fig. 6-17. They include ordinates between joints, and as an example of calculation of such ordinates, the computation for AN is given below for the influence line for

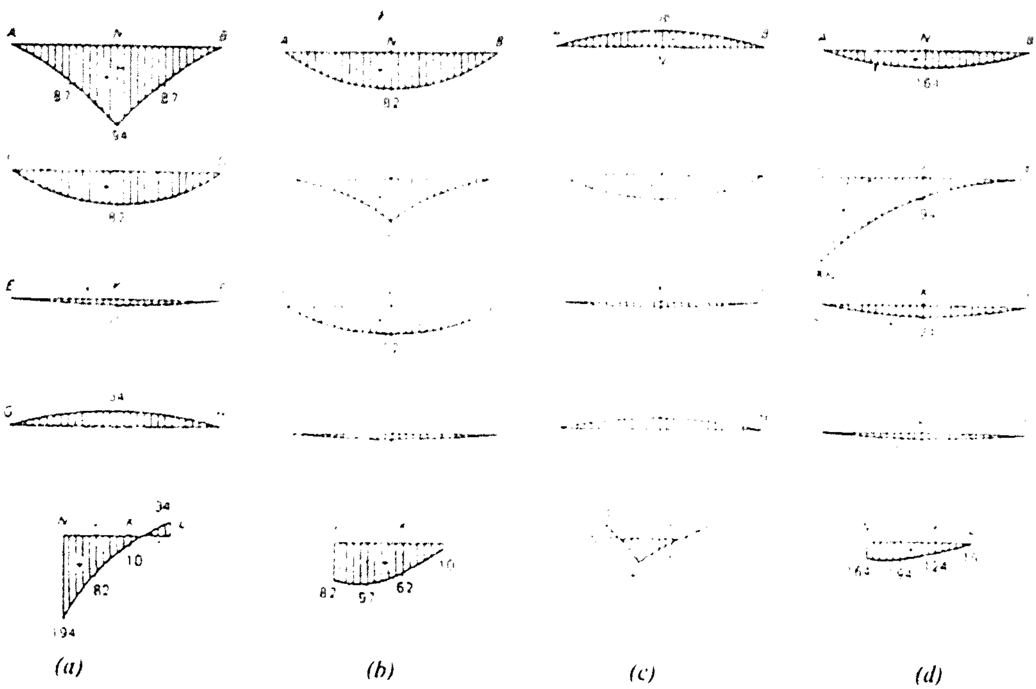


Fig. 6-17. Influence lines for the grid in Example 6-3. (a) Influence line for bending moment in main girder at center of AB, multiplier = *l* / 1000. (b)Influence line for bending moment in main girder at center of CD, multiplier = *l* / 1000. (c) Influence line for bending moment in cross-girder at joint J, multiplier = *l* / 1000.(d) Influence line for reaction at C, multiplier = *l* / 1000

the bending moment in Fig. 6-17a. The end-moment at N of member AN corresponding to the deflected shape in Fig. 6-17a is

$$M_{NA} = 12\frac{EI}{l} - \left[\frac{3E(4I)}{(l/2)^2}\right]0.194l = 2.688\frac{EI}{l}$$

The first term in this equation is the end-moment when the vertical joint displacements are restrained, and the second term is the end-moment caused by the vertical displacement. The bending moments caused by unit values of the vertical displacement are tabulated in Appendix E. The deflection measured from the straight line between A and N can be calculated by Eq. 6-5 or by the use of Appendix I. Therefore, the equation of the influence line between A and N is

$$\eta = l\,[\,0.194\,\varepsilon - 0.028\,(\varepsilon - \varepsilon^3)]$$

where $\varepsilon = (2x/l)$, and x is the distance from A to the desired point on AN.

6.8 Influence Lines for Arches and Trusses

6.8.1 General Superposition Equation

The concept of adding influence coefficients for statically determinate and statically indeterminate cases in Eq. 6.7 to obtain the influence coefficient for bending moment at a section of a straight member will be extended now for any action in a statically indeterminate structure.

The influence coefficients for any action in a linearly elastic statically indeterminate structure can be obtained by adding the influence coefficients for the same action in a released structure and the influence coefficients for the redundants multiplied by the values of the action due to unit values of the redundants. Let p be the number of influence coefficients to be calculated for any action of a structure statically indeterminate to the nth degree. If a unit point load is applied at j, one of the p locations where the influence coefficients are required, the influence coefficient $\eta_j = A_j$ is the value of the action in the statically indeterminate structure determined by the superposition equation

$$A_j = A_{sj} + [F_{1j} F_{2j} \ldots F_{nj}] \{A_u\} \qquad (6.13)$$

where $A_{sj} = \eta_{sj}$ is the value of the action due to a unit load at j in a released structure, $F_{ij} = \eta_{Fij}$ is the value of the ith redundant due to a unit load at j, and the elements of $\{A_u\}$ are the values of the action considered due to unit values of the redundants on the released structure.

If the statically indeterminate structure is subjected to a unit load acting separately at each of the p locations and Eq. 6-13 is applied, we obtain the following equation of superposition of influence coefficients:

$$\{\eta\}_{p \times 1} = \{\eta_s\}_{p \times 1} + [\{\eta_{F1}\} \vdots \{\eta_{F2}\} \vdots --- \vdots \{\eta_{Fn}\}]_{p \times n} \{A_u\}_{n \times 1} \qquad (6\text{-}14)$$

in which the elements of the submatrices $\{\eta_{Fi}\}$ are the p influence coefficients of the redundants F_i.

To use Eq. 6.14 the influence lines for the redundants must first be determined. These can be obtained by an analysis for p locations of the unit load. For each position, the n redundants are determined, thus giving one of the p ordinates of the influence line for each redundant.

Influence lines for the redundants can also be determined by direct application of Muller-Breslau principle.

The use of Eq. 6-14 for arches and trusses will be considered in the following two sections.

6-9 Influence Lines for Arches

Influence lines are very useful in the analysis of arch bridges. Here the load is applied through vertical members supporting the deck (Fig. 6-18). Let us consider the influence lines due to a unit vertical load in any position E on CD. This load is assumed to be transmitted to the arch at F vertically below E.

The fixed arch in Fig. 6-18a is statically indeterminate to the third degree. The simply supported arch in Fig. 6-18b is chosen as the released

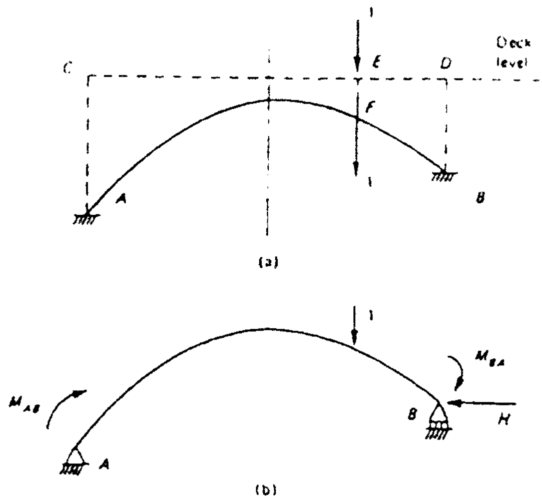


Fig. 6-18 Fixed arch and a released structure. (a) Arch supporting a bridge deck. (b) Statically determinate released structure.

structure with the clockwise end-moments M_{AB} and M_{BA}, and the inward horizontal force H as the redundants. The influence lines for the three redundants can be obtained by application of Muller-Breslau principle. For the influence line for M_{AB} an anticlockwise unit rotation is introduced at end A: the resulting vertical displacements of the arch axis are the influence ordinates. Similarly, for the influence line for M_{BA}, we introduce a unit anticlockwise unit rotation at B, and for the influence line for H a unit horizontal displacement is introduced outwards at either A or B.

The bending moment M corresponding to these end displacement can be conveniently obtained by column analogy, and hence the corresponding vertical deflection is calculated. The method of elastic weights may be used for this purpose. This procedure ignores the effect of axial deformation of the arch. In extremely flat arches, however, the axial deformations may have some effect, and this can be included as a correction using the approximate method.

The steps outlined above for finding the influence lines for the redundants are simple, but writing general algebraic expressions for the terms involved in the solution often leads to involved integrals. Simple expressions can be obtained only for a symmetrical parabolic arch in which the flexural rigidity EI is assumed to vary as the secant of the inclination, of the arch axis, Fig 6-19a. The properties of the analogous column for such an arch are given in Appendix J. The equations of the influence lines of the three redundants in such an arch (Fig. 6-19b) are

$$\eta_H = \frac{15l}{64h}(1-\varepsilon^2)^2 \qquad (6\text{-}15)$$

$$\eta_{MAB} = -\frac{l}{32}(1-\varepsilon)^2(5\varepsilon^2+6\varepsilon+1) \qquad (6\text{-}16)$$

$$\eta_{MBA} = \frac{l}{32}(1+\varepsilon)^2(5\varepsilon^2-6\varepsilon+1) \qquad (6\text{-}17)$$

where $\varepsilon = x/0.5l$, l is the span, and h is the rise. Equations 6-15 and 6-16 are plotted in Figs. 6-19c and d.

Tables of influence coefficients for parabolic, circular, or semielliptical prismatic and nonprismatic arches are available

The influence line for the stress resultant at any section of the arch in Fig. 6-19a can be determined by Eq. 6-14. For example, to obtain the influence line of the bending moment M_C at the crown, we first find the values of this action due to unit values of the redundants on the released structure of Fig. 6-19b: $\{A_u\} = \{-h, \frac{1}{2}, -\frac{1}{2}\}$, in which the order of the redundants is H, M_{AB} and M_{BA}. The influence line η_s for the moment M_C in the released structure is formed by two straight segments (as for a simple beam):

$$\left.\begin{aligned} \eta_s &= \frac{l}{4}(1-\varepsilon) \quad \text{for } 0 \le \varepsilon \le 1 \\ \eta_s &= \frac{l}{4}(1+\varepsilon) \quad \text{for } 0 \ge \varepsilon \ge -1 \end{aligned}\right\} \qquad (6.18)$$

The influence ordinates η_{MC} are given by Eq. 6-14

$$\eta_{MC} = \eta_s + (-h\eta_H + \frac{1}{2}\eta_{MAB} - \frac{1}{2}\eta_{MBA})$$

The influence ordinates on the right-hand side of this equation are given by Eq. 6-15,6-18. The shape of the influence line for M_C is shown in Fig 6-19.

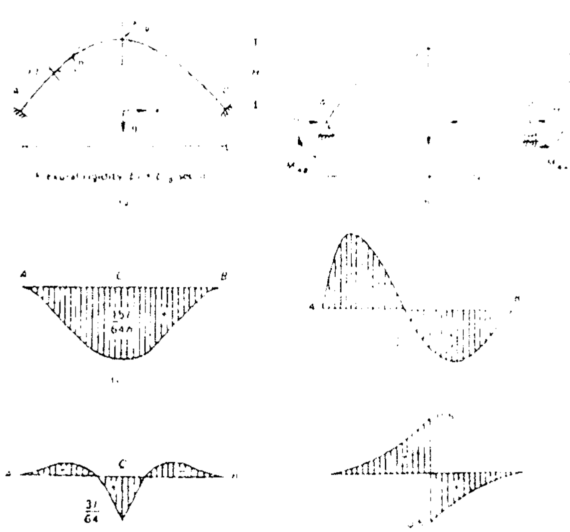


Fig 6-19 Influence lines for a parabolic arch with secant variation of flexual rigidity. (a) Parabolic arch with secant variation in EI. (b) Positive direction of the redundands. (c) Influence line for H. (d) Influence line for M_{AB}. (e) Influence line for M_C. (f) Influence line for V_C.

The influence line for shear V_C at the crown can be obtained in a similar manner; its shape is shown in Fig. 6-19

It can be easily shown that the horizontal thrust H due to a uniform load w per unit length of the horizontal projection of a parabolic arch, with any variation in EI and with hinged or fixed supports, is

$$H = \frac{ql^2}{8h} \tag{6-19}$$

and that the bending moment and the shear are zero at all sections. Since the area under an influence line is equal to the value of the action considered due to a uniform load q =1 (see Eq. 6-3), it follows that the area under η_H (Fig. 6-19c) is $l^2/8h$ and the area is zero under the other three influence lines in Fig 6-19.

6.10 Influence Lines for Trusses

Influence lines for the reactions or forces in the members of pin-connected trusses can be obtained by solving for several cases with the unit load at different joints. The influence lines can also be obtained from Eq. 6-14, which applies to any linearly elastic structure. To use this equation, we need the influence line for a statically determinate released truss, the influence lines for the redundants, and the values of the action due to unit values of the redundants. The procedure is illustrated by the following example.

Example 6-4 Find the influence line for the reaction at B and the forces in the members labeled Z_1 and Z_2 in the truss of Fig. 6-20a. The unit load can act at the nodes of the lower chord only. All the members are assumed to have the same value of l/aE, l being the length, and a the cross-sectional area of the members.

A released structure is shown in Fig 6-20b, in which the redundants F_1 and F_2 are taken as the forces in members Z_3 and Z_4 respectively. The influence ordinates η_s for the values of the required actions in the released structure are plotted in Figs. 6-20c, d, and e. These can be easily checked by simple statics.

According to Muller-Breslau principle, the influence line for F_1 can be obtained by cutting the member Z_3 and applying equal and opposite forces (causing compression in Z_3) to the remaining truss, so as to produce a relative unit displacement at the cut section. Then the deflected shape of the bottom chord of the truss gives the influence line for F_1. This is the same as finding the deformations in the actual structure due to a unit extension of member Z_3, such as that caused by a rise in temperature or a lack of fit in this member.

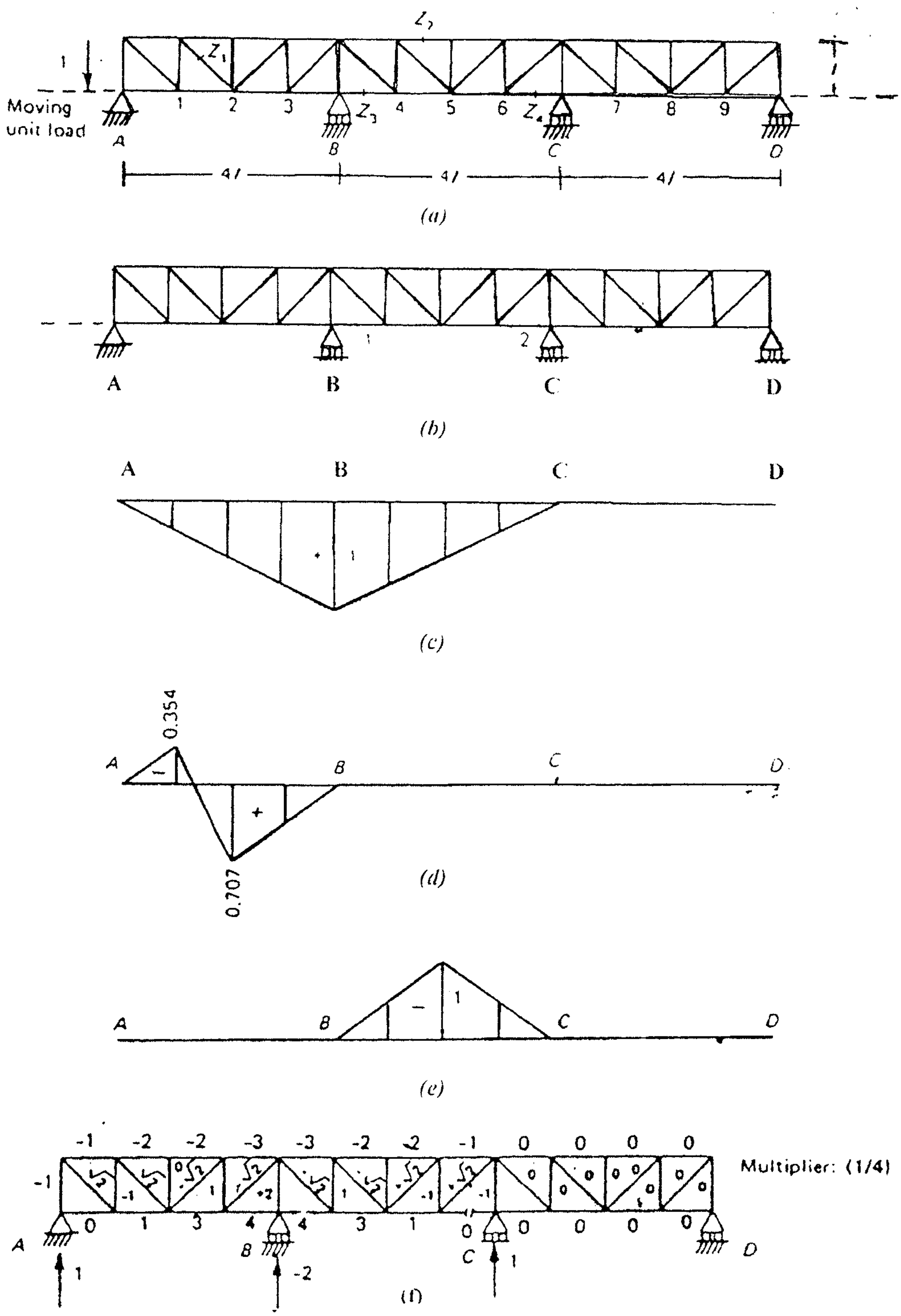


Fig. 6-20 Analysis of the continuous truss of Example 6-4. (a) Continuous truss. (b) Released structure and coordinate system. (c) Influence line for R_b in the released structure. (d) Influence line for the force in Z_1 in the released structure. (e)Influence line for the force in Z_2 in the released structure. (f) Reactions and forces in members due to $F_1 = 1$.

exibility matrix of the released structure is

$$[f] = \frac{1}{8Ea}\begin{bmatrix} 57 & 3 \\ 3 & 57 \end{bmatrix}$$

ements of this matrix can be easily checked by virtual work. For convenience, the forces members due to $F_1 = 1$ are shown in Fig. 6-20f. Making use of symmetry of the ure, the forces in members due to $F_2 = 1$ can also be deduced from this figure.

A unit extension in member Z_1 produces the redundants $\{F_1, F_2\}$ given by Eq. 2-9

$$[f]\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

From this equation we can see that the redundants have the values $F_1 = -S_{11}$ and S_{21}, where S_{11} and S_{21} are the elements of the first column of the stiffness matrix [S] of eased structure. Similarly, the values of the redundants corresponding to a unit extension are equal to minus the elements in the second column of [S]. In the above example,

$$[S] = [f]^{-1} = \frac{Ea}{405l}\begin{bmatrix} 57 & -3 \\ -3 & 57 \end{bmatrix}$$

he values of the redundants $F_1 = -57Ea/405l$ and $F_2 = 3Ea/405l$, the forces in all the ers can be determined. The corresponding deflections at the joints 1, 2, ---, 9 on the chord give the influence ordinates of the redundant F_1. The influence line of the lant F_1 is plotted in Fig. 6-21a. Because the structure is symmetrical, the same ordinates rsed order are the ordinates at the nine joints of the influence line for F_2.

For the influence line of the reaction at B, we use Eq. 6-14 to determine the coordinates oints 1, 2, ---, 9:

$$\{\eta\}_{9\times1} = \{\eta_s\}_{9\times1} + [\{\eta_{u1}\} \vdots \{\eta_{u2}\}]_{9\times2}\{A_u\}_{2\times1} \qquad (6\text{-}20)$$

ements of $\{A_u\}$ are the reactions at B due to $F_1 = 1$ and $F_2 = 1$ (see Fig. 6-20f)

$$\{A_u\} = \begin{Bmatrix} 0.5 \\ 0.25 \end{Bmatrix}$$

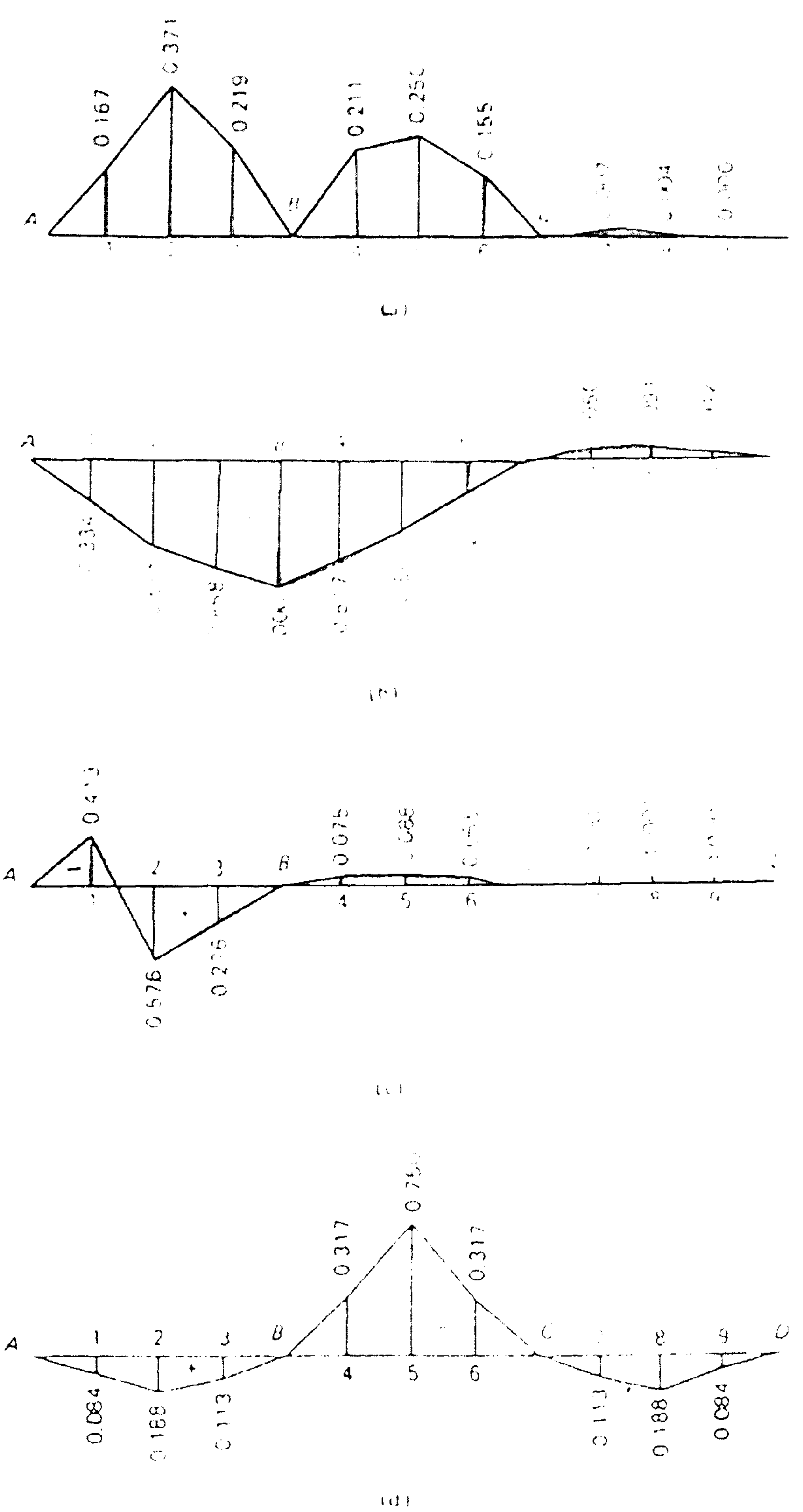


Fig 6-21 Influence lines for the continuous truss of Fig. 6-20 (Example 6-4). (a) Influence line for redundant F_1 (force in member Z_3). (b) Influence line for vertical reaction at B. (c) Influence line for force in member Z_1. (d) Influence line for force in member Z_2.

An upward reaction is considered positive. Substituting in Eq. 6-14, we obtain the influence ordinates for R_B.

$$\{\eta\} = \begin{Bmatrix} 0.250 \\ 0.500 \\ 0.750 \\ 0.750 \\ 0.500 \\ 0.250 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} -0.167 & 0.000 \\ -0.371 & -0.004 \\ -0.219 & -0.007 \\ -0.211 & -0.155 \\ -0.250 & -0.250 \\ -0.155 & -0.211 \\ -0.007 & -0.219 \\ -0.004 & -0.371 \\ 0.000 & -0.167 \end{bmatrix} \begin{Bmatrix} -0.5 \\ 0.25 \end{Bmatrix} \begin{Bmatrix} 0.334 \\ 0.685 \\ 0.858 \\ 0.817 \\ 0.562 \\ 0.275 \\ -0.050 \\ -0.091 \\ -0.042 \end{Bmatrix}$$

The influence line for R_B is plotted in Fig. 6.21b. The influence lines for the forces in members Z_1 and Z_2 are determined by Eq. 6.14 in the same way, and the results are plotted in Figs 6-21c and d.